Automated Market Makers: Towards a Microfounded Theory

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Automated Market Makers

- Automated Market Makers (AMMs) are the dominant paradigm for Decentralized Exchanges (DEXs)

- But what are they?
  - ❑ What are their fundamental properties?
  - ❑ How should they be designed?

Roadmap to a Theory of AMMs

1. Design Space:
    - Construct methodology and formal language for the description of AMMs
2. Economic Model:
    - Identify tradeoffs and characterize market equilibrium
3. Mechanism Design:
    - Propose well-defined objectives functions
    - Characterize optimal design of AMMs (optimal slippage and fees)

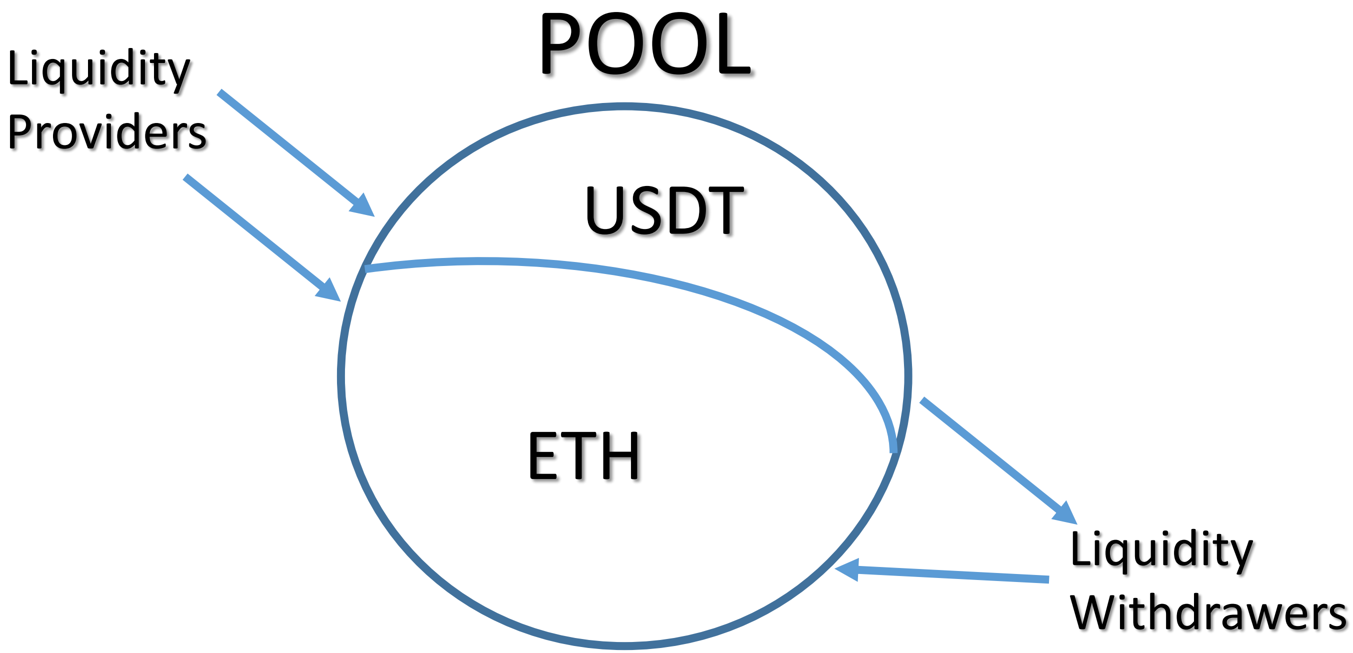
The microeconomics of AMMs

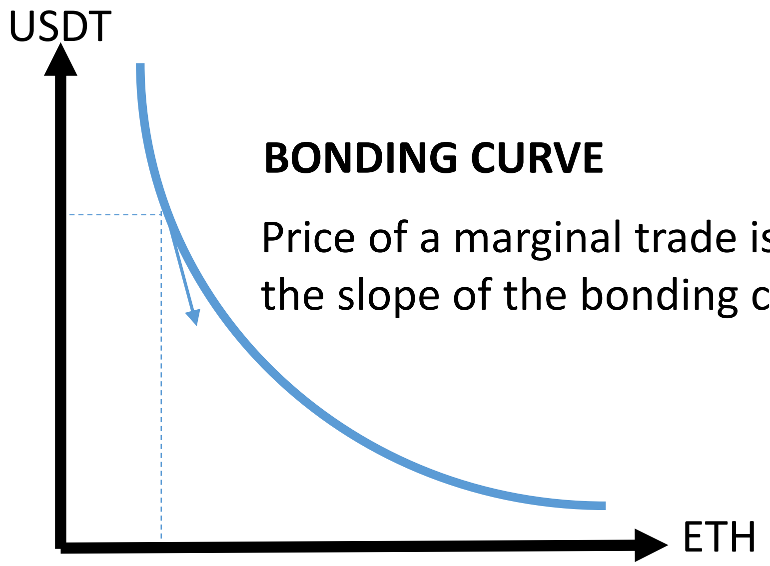
- Show that standard microeconomics is *the* right language to describe the design space of AMMs

- Use convex optimization and economically interpret our findings:
    1. Arbitrageurs solve compensated demand problem
    2. Dual problem is more intuitive and powerful than primal

AMMs are two-sided markets

Uniswap

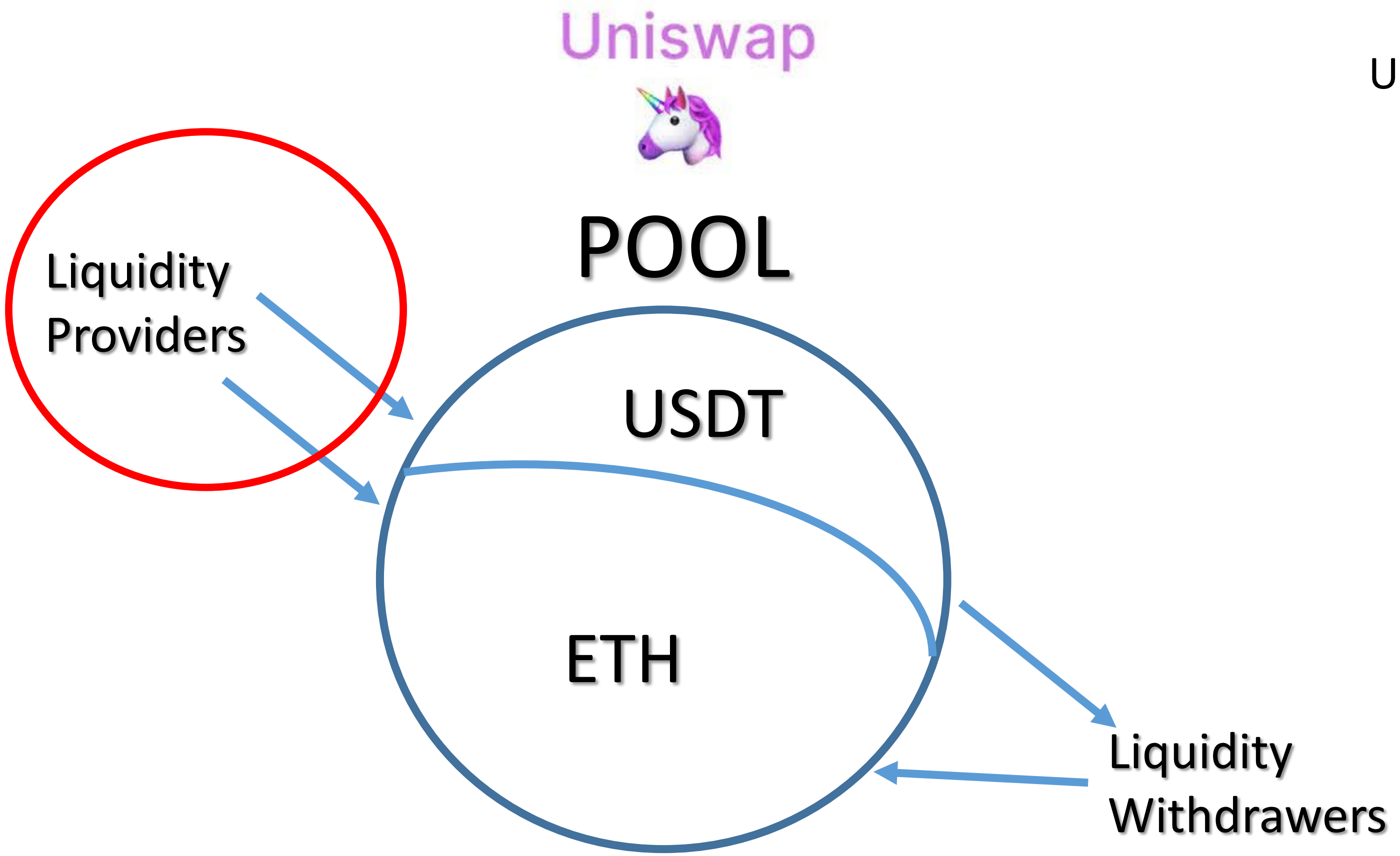


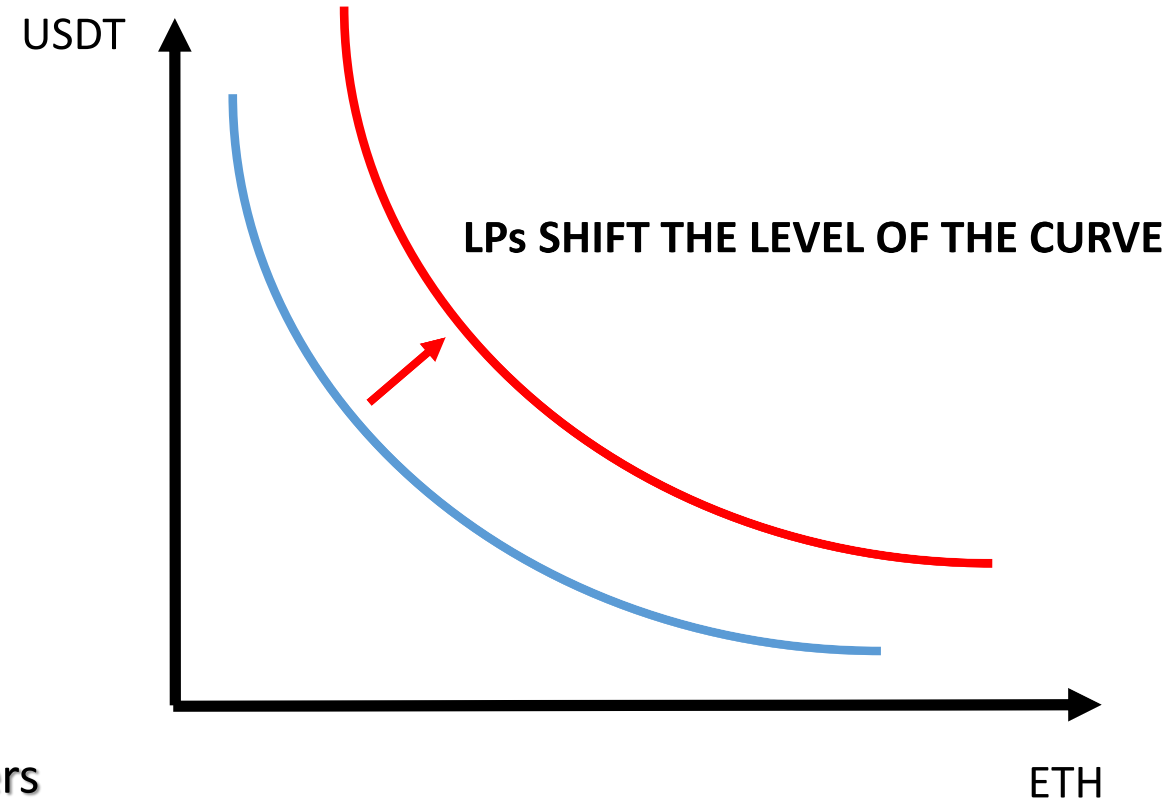


BONDING CURVE

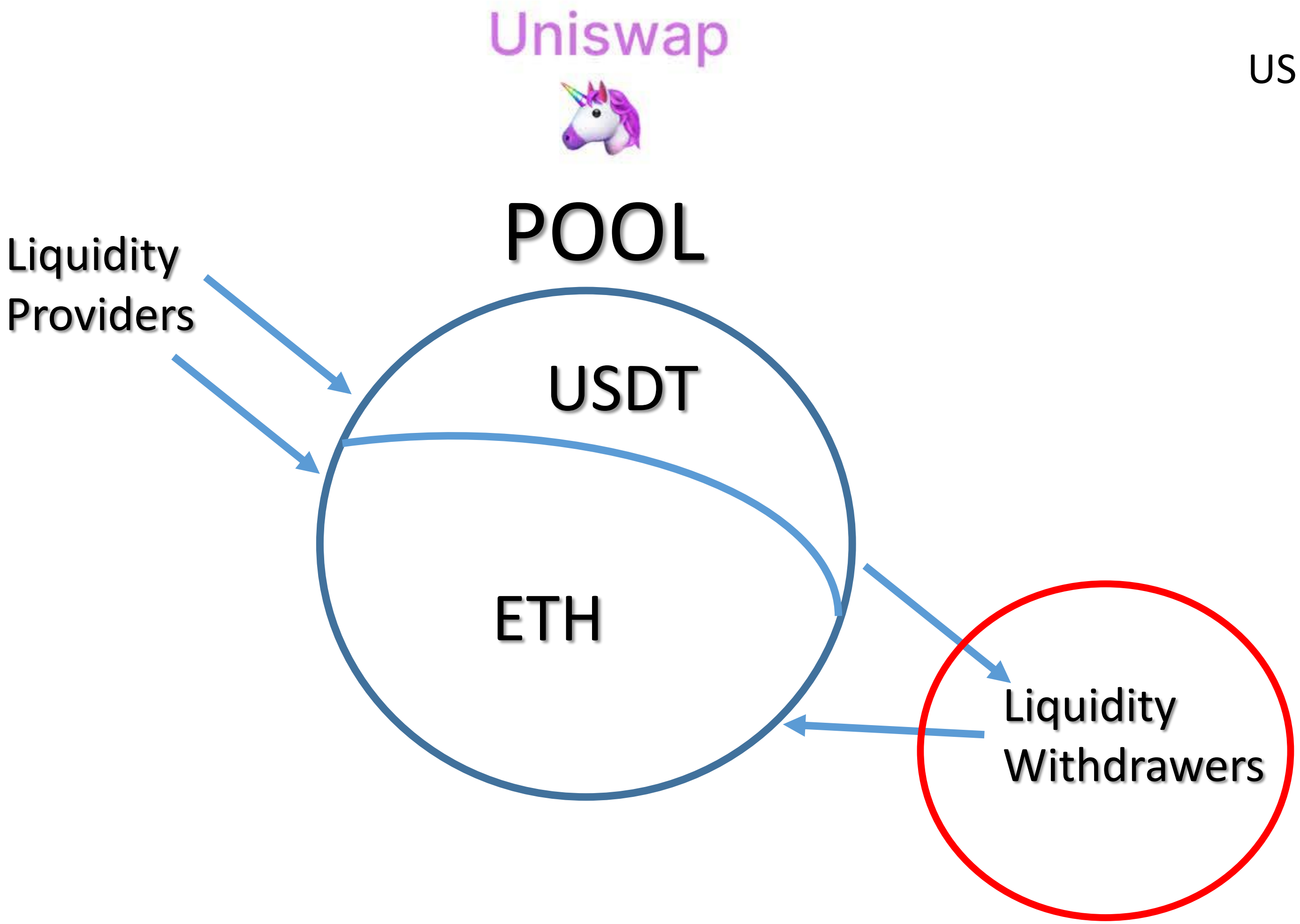
Price of a marginal trade is given by the slope of the bonding curve.

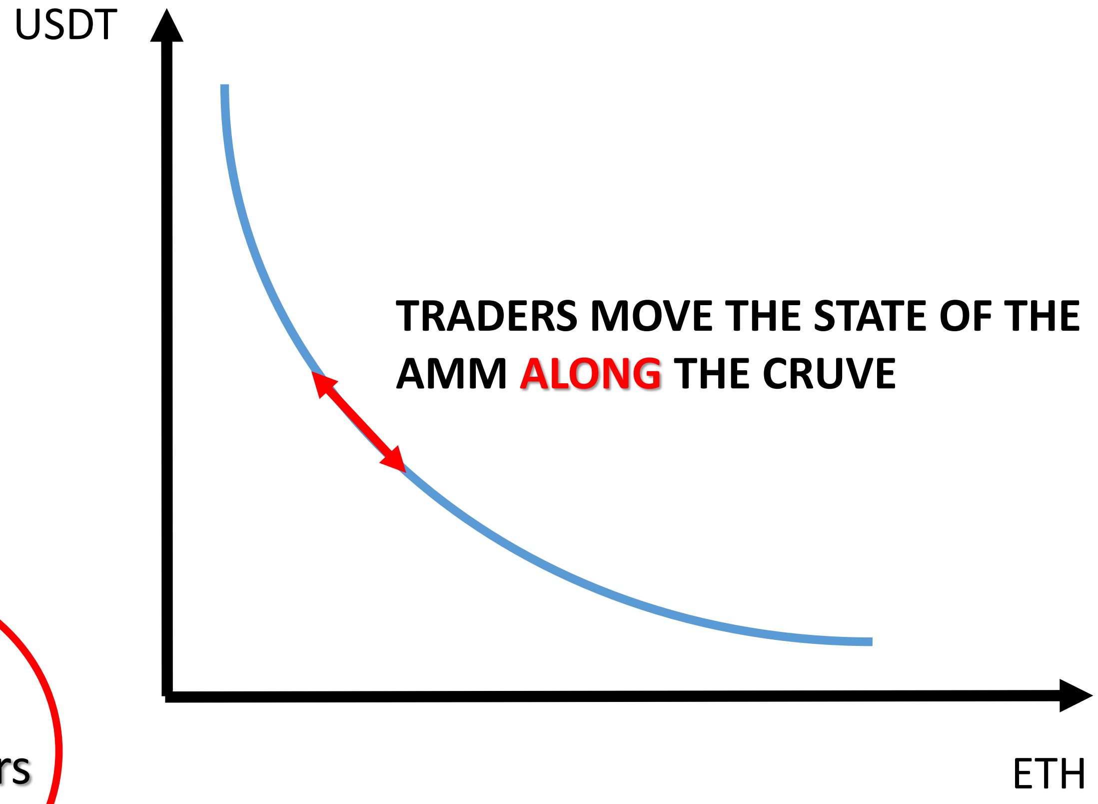
Impact of Liquidity Providers





Impact of Traders





Constant Function Market Makers

- Most AMMs are Constant Function Market Makers (CFMMs)

- A trade $T = (I, 0)$ is admissible iff

$$U(R + I - O) \geq U(R)$$

  - $U: R_+^n \rightarrow R$ is the trading function
  - $R \in R_+^n$ is the level of reserves
  - $I, O \in R_+^n$ are the reserves input and output from the trade

- **Uniswap**: $R \in R_+^2$ and $U = R_1 R_2$

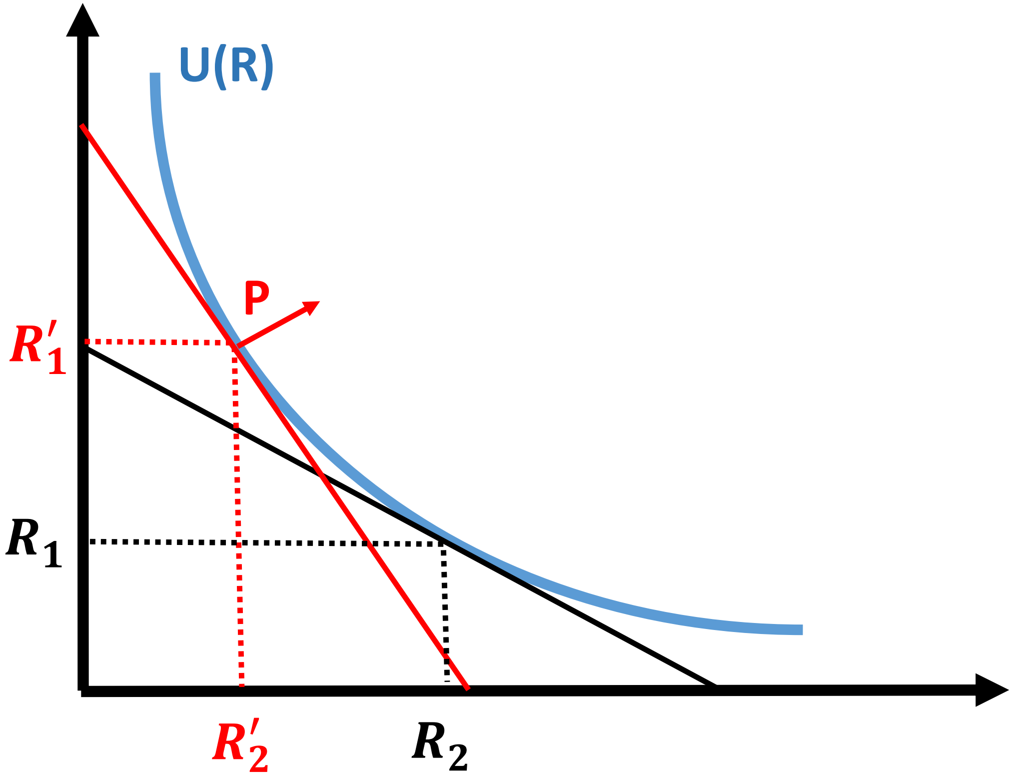
Arbitrageur Problem

- Arbitrageur observes a reference price **P** for the assets

- She solves the following problem

$$Min_{R'}\, P^T R'$$
$$s.t.\, U(R') \geq \overline{U} = U(R)$$

- Identical to **Hycksian demand**!



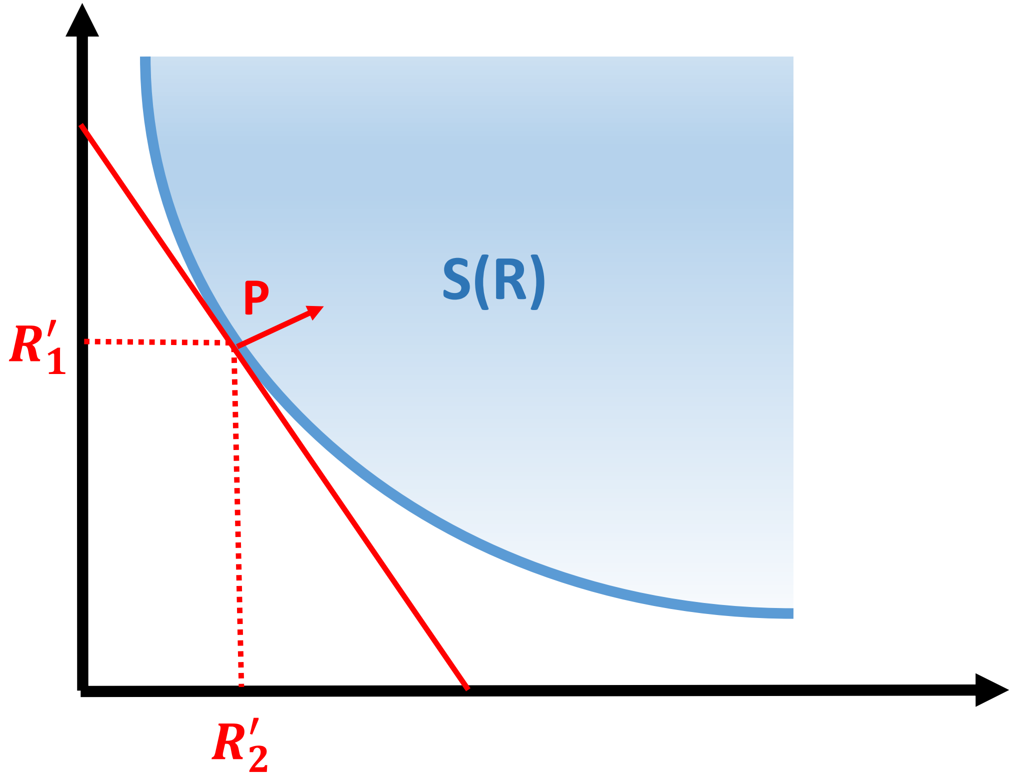
Oracle Property

- If the trading set **S(R)** is strictly convex, there exists a unique no-arbitrage trade $\boldsymbol{R'^*}$

$$P = \lambda \nabla U(R'^*)$$

where λ is a scaling factor

- **<u>Oracle Property</u>**: Arbitrageurs synchronize off-chain and on-chain prices when trading set is strictly convex



CFMM Equivalence

- Two CFMMs with the same trading set are equivalent

- Two CFMMs are equivalent iff their trading functions are monotonic transformations of one another, i.e. $U = f \circ U$ with f strictly increasing

- **Example**: Balancer $R \in R_+^2$ and $U = R_1^{1/2} R_2^{1/2} = \sqrt{R_1 R_2} = \sqrt{U'}$ where U' is the trading function of Uniswap

Expenditure Function

- Express problem in the dual space

- Expenditure function is the portfolio value of LPs in the absence of arbitrage opportunities

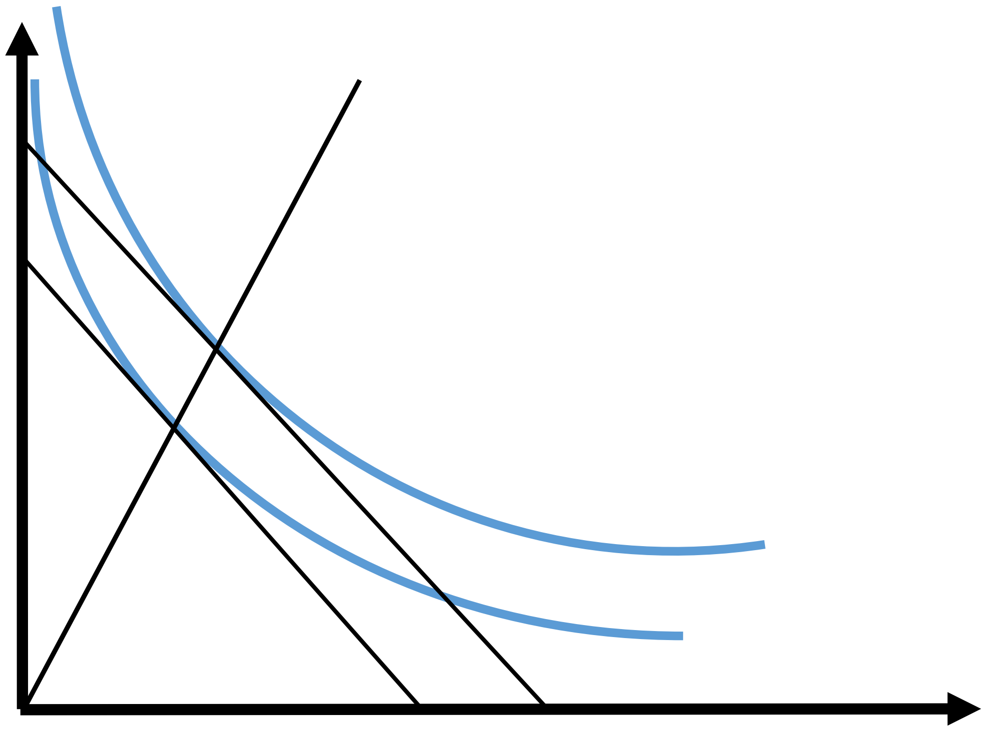
$$E(P,V) = min_R\{P^T R | U(R) \geq V\}$$

- Properties:
    1. Shephard's Lemma: $R^* = h(P,V) = \nabla_P E(P,V)$

    2. Separability: $E(P,V) = \varphi(V)e(P)$ iff the trading function is homothetic

Homothetic Trading Functions

- The no-arbitrage price is homogenous of degree zero in the liquidity of the AMM iff its trading function is homothetic
- An homothetic function is a monotonic transformation of a function that is homogenous of degree one
  - ➢ True for Uniswap, Balancer but not for Curve!
- If we want prices to be independent of overall liquidity, we can focus on trading functions that are homogenous of degree one



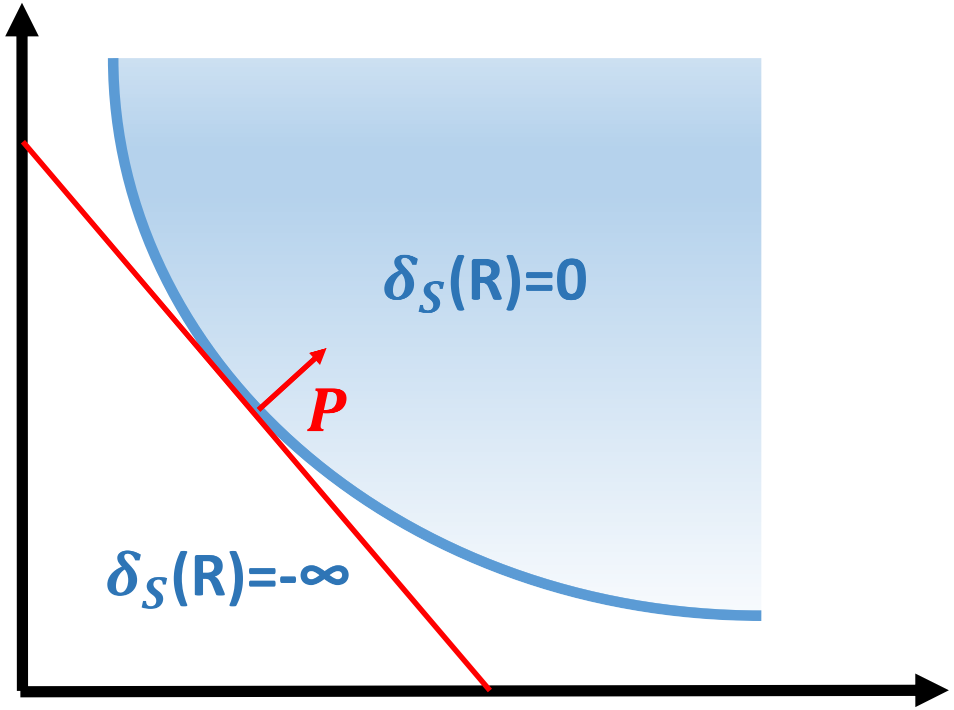
Duality

- The expenditure function is the conjugate of the indicator function $\delta_S(R)$

$$\mathrm{E}(\mathrm{P},\mathrm{V}) = \delta^*_{S(V)}(P) = -sup_R\{\delta_{S(V)}(R) - P^T R\}$$

- If S(V) is convex, the conjugate of the expenditure function is the indicator function

$$\delta_{S(V)}(R) = \delta^{**}_{S(V)}(R)$$
$$= -sup_P\{\delta^*_{S(V)}(P) - P^T R\}$$



Optimal design

- Design problem is more intuitive in the dual space

- Given a portfolio value function, we can search for the CFMM that generates it!

- **Example**: Uniswap expenditure function $E(P,V) = 2\sqrt{VP_1P_2}$

  Its conjugate yields the indicator function of its trading set

$$\delta_{S(V)}^{**}(R) = -sup_P\{E(P,V) - P^T R\}$$
$$= \begin{cases} 0 \; if \; R_1R_2 \geq V \\ -\infty \; otherwise \end{cases}$$

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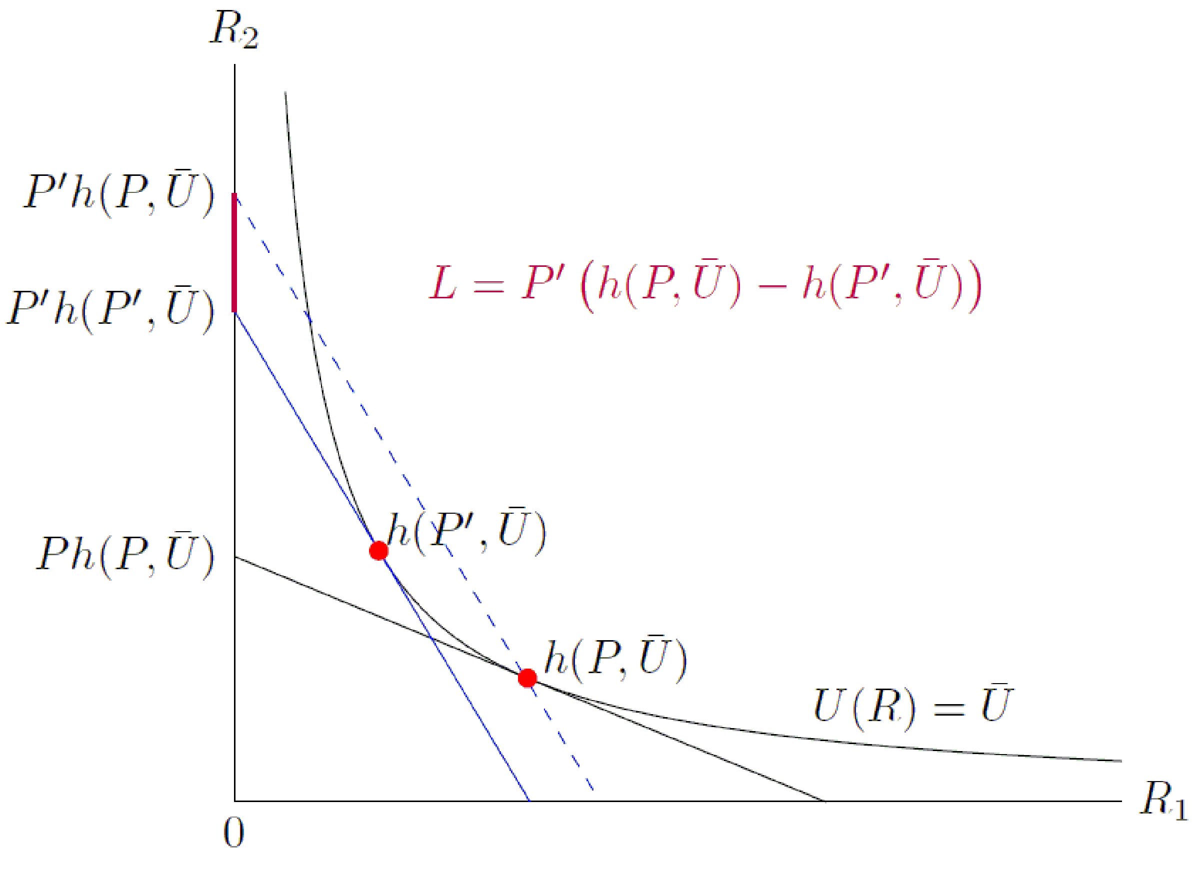
Optimal design

- Method can be applied to arbitrary payoffs
- For example, what is the CFMM that generates the payoffs of a European Option?
  - Answer: Compute the conjugate function of the Black-Scholes formula (Angeris et al. 2021)!
  - Useful? Could solve the oracle problem for options vaults

Impermanent Loss

- Another application of duality is the computation of impermanent losses

- The impermanent loss is the difference between the portfolio value of the AMM and of a static position after a price change

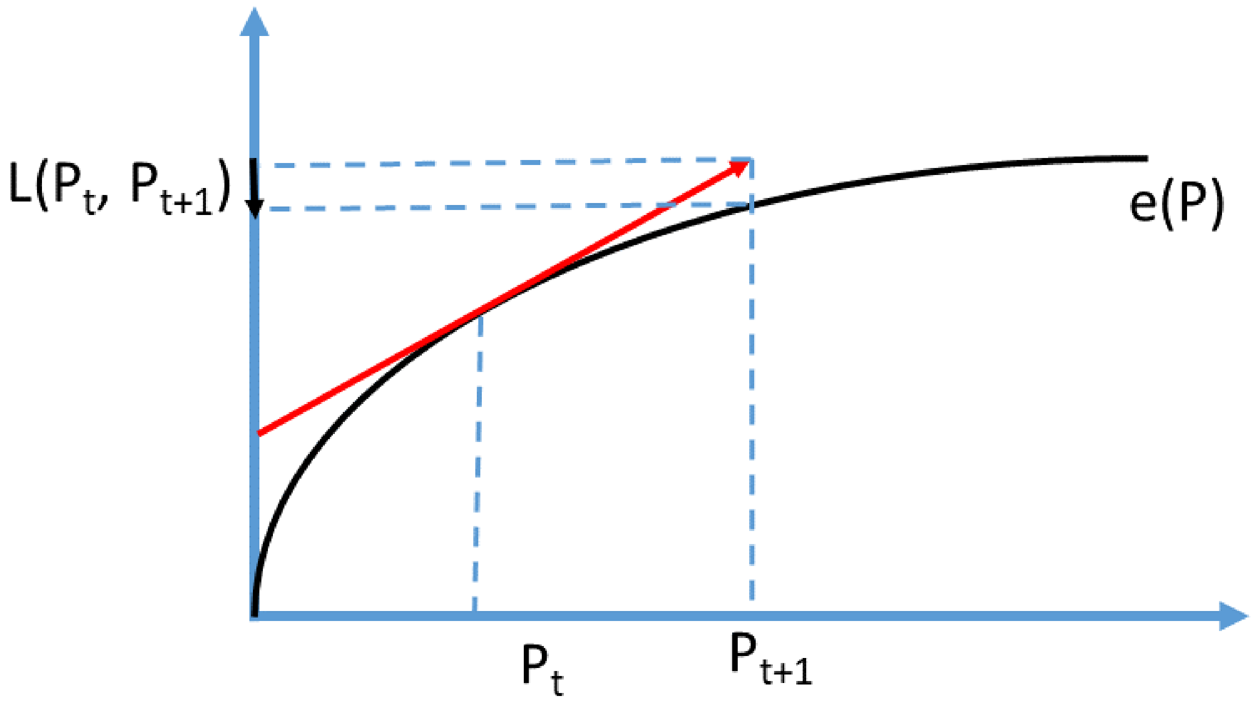


(a) P increases

Impermanent Loss in Dual Space

- Much simpler to compute impermanent losses in the dual space
- All the information is encapsulated in the Expenditure function:

$$L(P, P'; V) = E(P'; V) - P'^T \nabla E(P; V)$$



Conclusion

- Standard microeconomics is the natural language to establish the properties of CFMMs
- Formulation in the dual space is more powerful and more intuitive
- Now that we have formalized the design space, next task is to identify the economic tradeoffs
- For that, we need to turn our attention to the problem solved by the LPs
- Material for another presentation!