THALES

Building a future we can all trust

Frédéric BARBARESCO
THALES – 21/05/26
Séminaire ARISTOTE

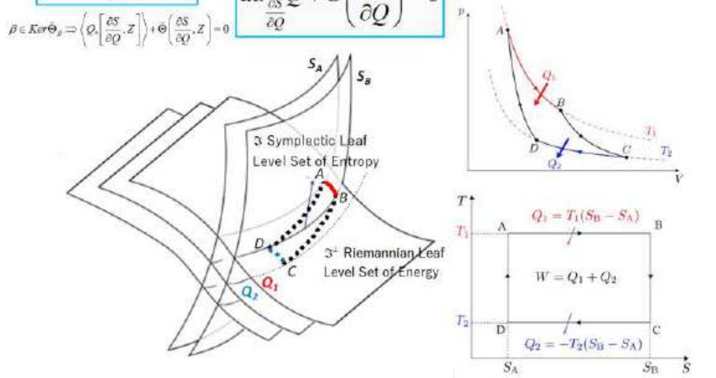
IA informée par les structures symplectiques de la géométrie de l'information et la « thermodynamique des groupes de Lie »

ENTROPY DEFINITION: Entropy is Invariant Casimir Function on Leaves of Symplectic Foliation associated to Coadjoint Orbits generated via Moment Map of Symmetry Group acting on the System.

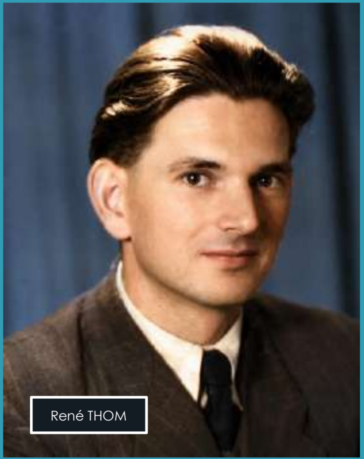
$$\boxed{\{S,H\}_{\tilde{\Theta}}(Q)=0}$$

$$\beta\in Ker\tilde{\Theta}_{\beta}\Rightarrow\left\langle Q,\left[\frac{\partial S}{\partial Q},Z\right]\right\rangle+\tilde{\Theta}\left(\frac{\partial S}{\partial Q},Z\right)=0$$

$$\boxed{ad^{*}_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0}$$



> *« Au départ, la théorie de la stabilité structurelle m'avait paru d'une telle ampleur et d'une telle généralité, qu'avec elle je pouvais espérer en quelque sorte remplacer la thermodynamique par la géométrie, géométriser en un certain sens la thermodynamique, éliminer des considérations thermodynamiques tous les aspects à caractère mesurable et stochastiques pour ne conserver que la caractérisation géométrique correspondante des attracteurs. »*
>
> **René THOM**



René THOM

> *« Il n'y a rien de plus dans les théories physiques que les groupes de symétrie si ce n'est la construction mathématique qui permet précisément de montrer qu'il n'y a rien de plus » -*
>
> **Jean-Marie Souriau**

Jean-Marie SOURIAU

Projet de Panthéonisation de Sadi Carnot – Ingénieur-Savant inventeur de la thermodynamique et du 2nd principe de la thermodynamique

Ils soutiennent l'entrée au Panthéon de Nicolas Léonard Sadi Carnot, le physicien, 1796-1832

Une mobilisation nationale pour la science, l'ingénierie et la jeunesse

Excellence scientifique

Formation et transmission

Industrie et transition énergétique

Elus de la République

François Patriat

Paul Midy

Mairie de Nolay (21)

Mairie de Cerny (91)

Signataires

Philippe Aghion, Professeur au Collège de France, Prix Nobel d'Économie 2025.

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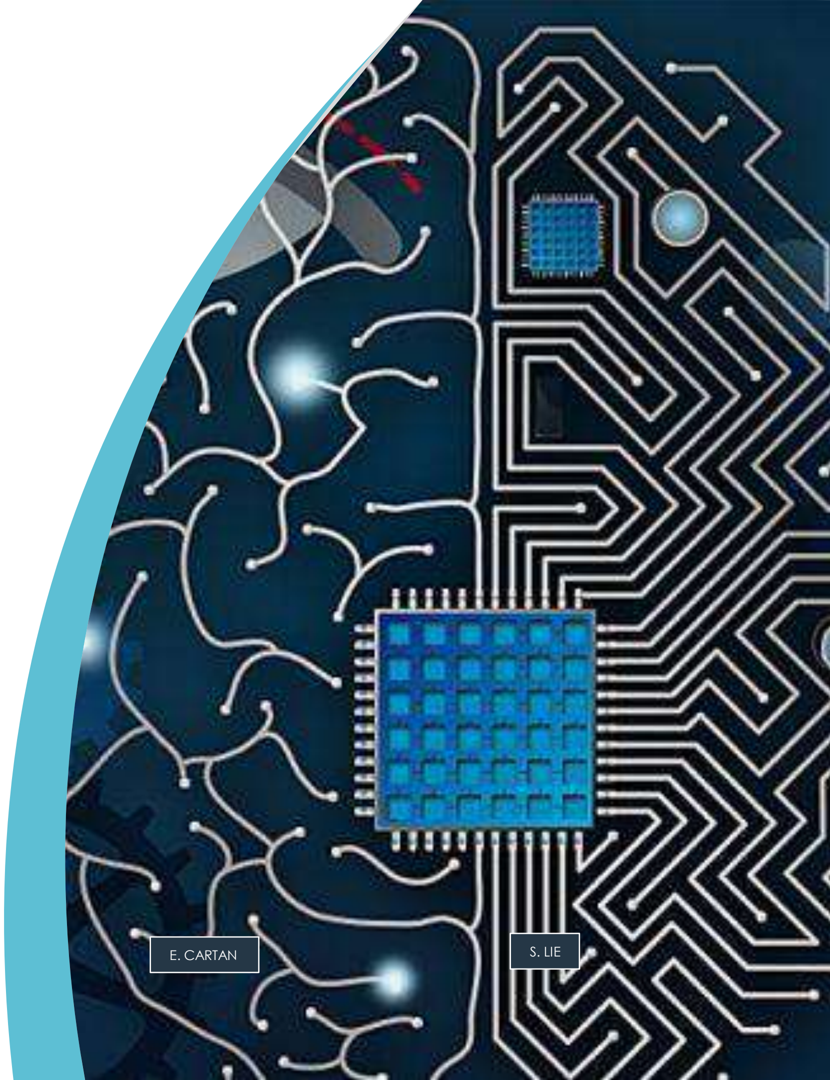
Daniel Rouan, Vice-président de la Société Française de Physique (SFP)

Leï Sun, Co-fondatrice de ChinaPavillon

https://sadi-carnot-physicien.fr/

THALES

From Empirical Science to Artificial Science Intelligence



Classical Computing
GPU & HPC-GPU

… DEADLOCK

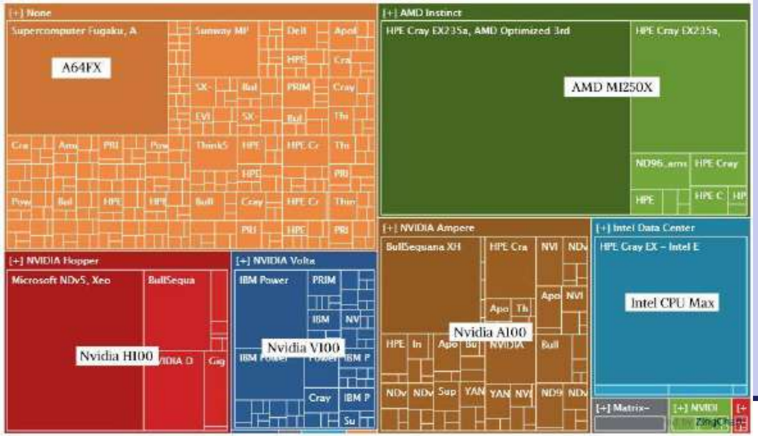
Quantum Computing
QPU & HPC-QPU

…NEW ADVANTAGES

Computing power has fostered AI emergence:

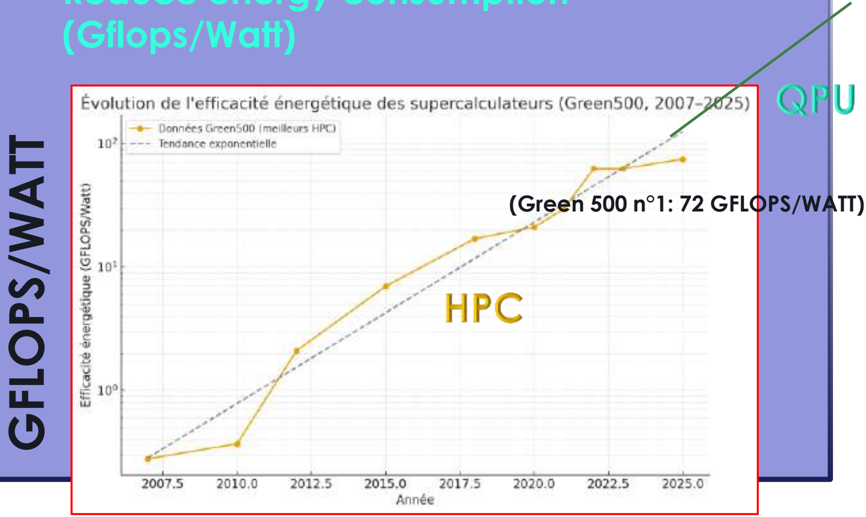
- **Hundreds of GPUs for Deep Learning**
- **Tens of thousands of GPUs for LLMs**

Growth is slowing down due to energy constraints



Quantum computing allows to:

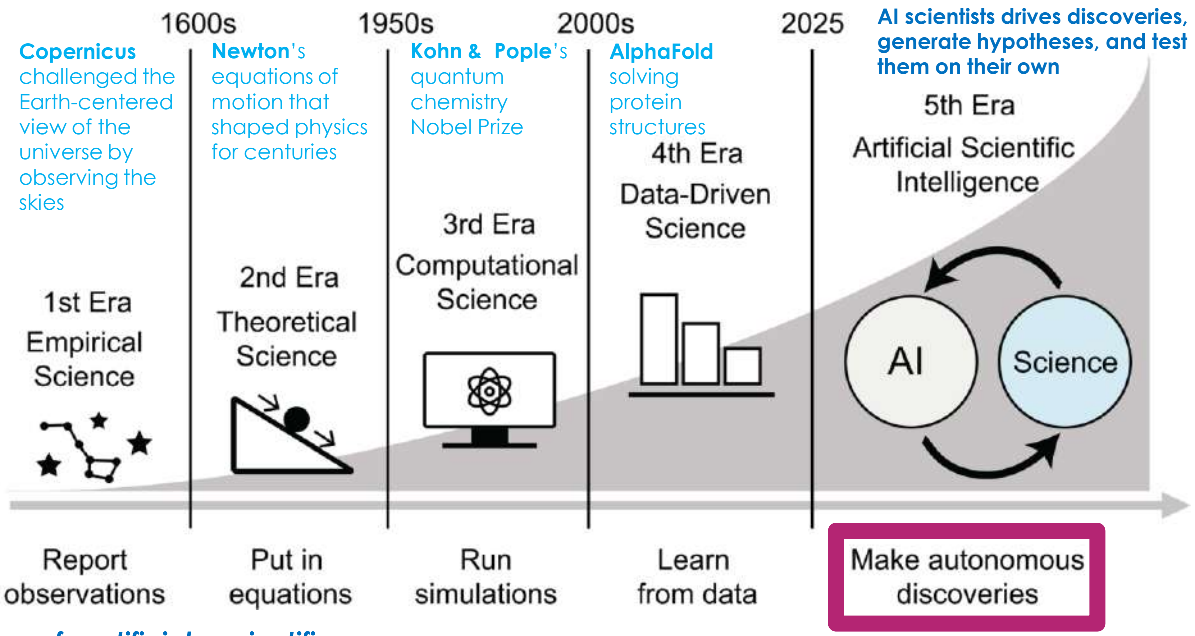
- **Exponentially improve computing power**
- **Reduce energy consumption (Gflops/Watt)**



(Green 500 n°1: 72 GFLOPS/WATT)

FROM EMPIRICAL SCIENCE TO ARTIFICIAL SCIENTIFIC INTELLIGENCE

The fifth era of science: Artificial scientific intelligence



1600s | 1950s | 2000s | 2025

Copernicus challenged the Earth-centered view of the universe by observing the skies

1st Era
Empirical Science

Report observations

Newton's equations of motion that shaped physics for centuries

2nd Era
Theoretical Science

Put in equations

Kohn & Pople's quantum chemistry Nobel Prize

3rd Era
Computational Science

Run simulations

AlphaFold solving protein structures

4th Era
Data-Driven Science

Learn from data

AI scientists drives discoveries, generate hypotheses, and test them on their own

5th Era
Artificial Scientific Intelligence

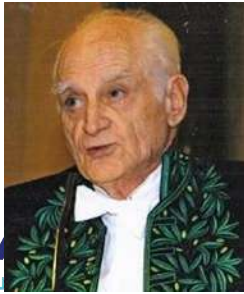
Make autonomous discoveries

NINA MIOLANE
Ecole Polytechnique, Imperial College, PhD INRIA, Post-Doc Stanford, UC Santa Barbara
L'Oréal-UNESCO Fellowship for Women In Science

Miolane N (2025) **The fifth era of science: Artificial scientific intelligence**. PLoS Biol 23(6): e3003230. https://doi.org/10.1371/journal.pbio.3003230

"The era of artificial scientific intelligence is here. As algorithms generate discoveries at scale, what role remains for ~~human scientists~~ Engineer?"

« ce qui distingue l'être humain, c'est qu'il est par essence « déprogrammé » et même non programmable : il n'est donc prisonnier d'aucun code lui assignant une fonction donnée et peut en permanence mobiliser en toute indépendance la totalité des capacités qui se refusent à une quelconque externalisation, et qu'il peut de ce fait appliquer à sa guise à la réalisation des fins qu'ils se donne à lui-même.. » - Michel SERRES (Académie Française)



https://www.academie-francaise.fr/actualites/communication-de-m-michel-serres-0

AI FOR SCIENCE & ENGINEERING – THE DISRUPTION IN MATHS, PHYSICS, …, ENGINEERING

French Academy of Science:
AI for Sciences

- https://www.academie-sciences.fr/lintelligence-artificielle-pour-les-sciences

Intelligence Artificielle pour les Sciences

17 Mars 2026, Académie des Sciences

Les mathématiques à l'heure de l'IA

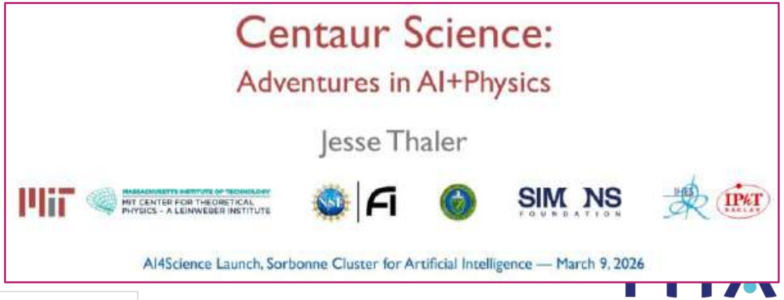
Gérard Biau
Académie des sciences, mars 2026

SCAI Sorbonne University:
AI4Science : Towards « Centaur Science »

- https://ai4science-su.sciencesconf.org/

AI4SCIENCE

Launch Day, March 9th, 2026



Centaur Science:
Adventures in AI+Physics

Jesse Thaler

AI4Science Launch, Sorbonne Cluster for Artificial Intelligence — March 9, 2026

PHYSICAL AI & AI FOR SCIENCE

"students should prioritize physics and mathematics over coding." - Elon Musk and NVIDIA CEO Jensen Huang

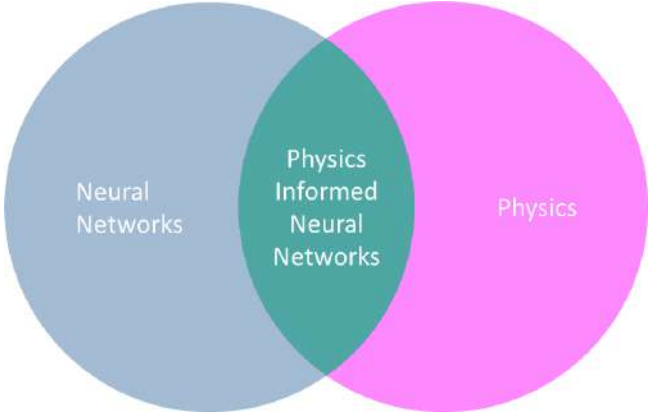
U.S. DEPARTMENT of ENERGY
Genesis Mission

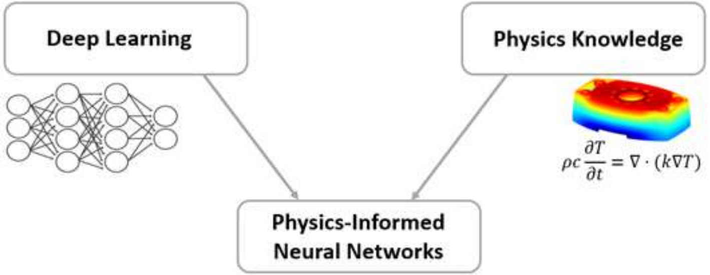
USA DoE GENESIS MISSION

A National Mission to Accelerate Science Through AI, to build the world's most powerful scientific platform to accelerate discovery science, strengthen national security, and drive energy innovation.

NVIDIA COSMOS & PhysicsNeMo

NVIDIA Cosmos™ Transfer & Predict enable physically based synthetic data generation in simulation for physical AI. PhysicsNeMo is an open-source Python framework for building, training, and fine-tuning physics AI models at scale.





$$\rho c \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T)$$

PROJECT PROMETHEUS

Jeff Bezos launches Project Prometheus, a physical AI venture with 6.2 billions $, physical AI is emerging now across factories and aerospace

NEURAL CONCEPT

Advanced Computational Large Eddy Simulation by using Physics-Informed deep learning techniques, engineers can improve the accuracy and make them faster and more accessible.

US DoE Genesis Mission

https://www.energy.gov/undersecretaryforscience/genesis-mission/genesis-mission

DoE has identified an initial 26 national science and technology challenges to accelerate U.S. innovation and leadership. Among them:

- Reenvisioning Advanced Manufacturing and Industrial Productivity
- Recentering Microelectronics in America
- Securing U.S. Leadership in Data Centers Accelerating Materials Discovery, Production, and Qualification for Strategic Deterrence
- Achieving AI-Driven Autonomous Laboratories
- Designing Materials with Predictable Functionality
- Enhancing Particle Accelerators for Discovery
- Discovering Quantum Algorithms with AI
- Realizing Quantum Systems for Discovery

THE WHITE HOUSE
WASHINGTON

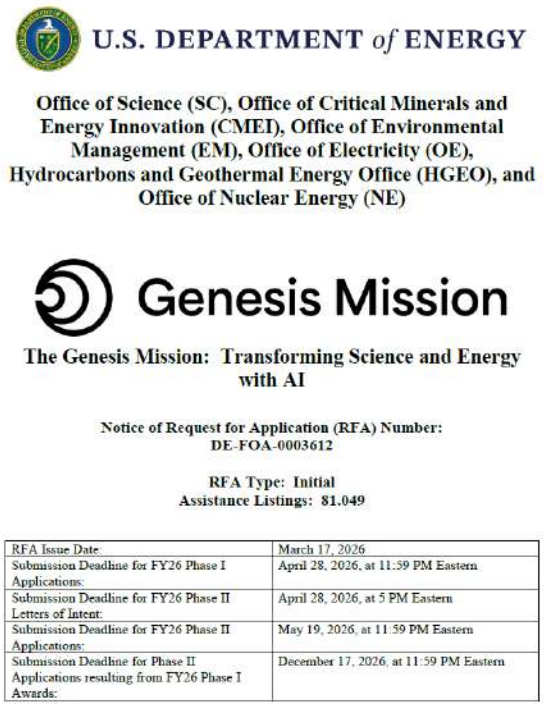
PRESIDENTIAL ACTIONS

LAUNCHING THE GENESIS MISSION

The Genesis Mission unites DOE National Labs, industry, academia, and more to harness AI for breakthroughs in energy dominance, discovery science, and national security.

Energy Department Announces $293 Million in Funding to Support Genesis Mission National Science and Technology Challenges

US DoE Genessis Mission: 293 M$ call including Physics-Informed AI (PINN)



U.S. DEPARTMENT *of* ENERGY

Office of Science (SC), Office of Critical Minerals and Energy Innovation (CMEI), Office of Environmental Management (EM), Office of Electricity (OE), Hydrocarbons and Geothermal Energy Office (HGEO), and Office of Nuclear Energy (NE)

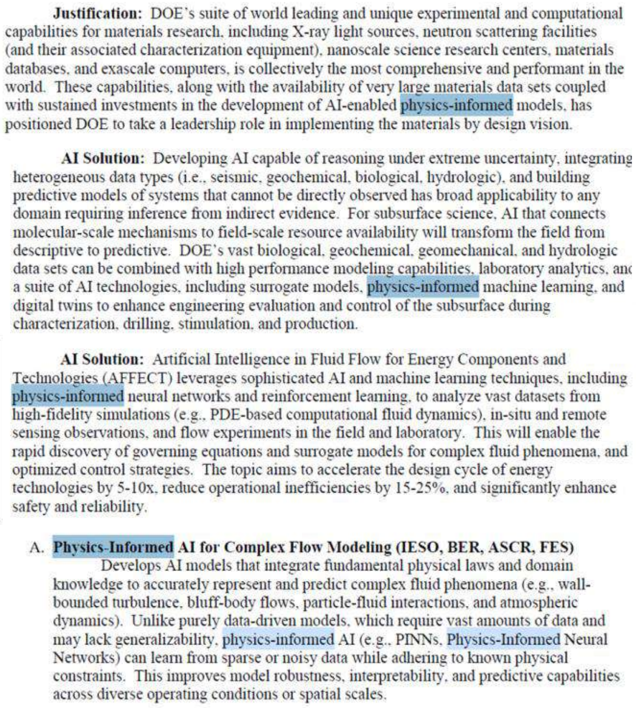
Genesis Mission

The Genesis Mission: Transforming Science and Energy with AI

Notice of Request for Application (RFA) Number:
DE-FOA-0003612

RFA Type: Initial
Assistance Listings: 81.049

RFA Issue Date:	March 17, 2026
Submission Deadline for FY26 Phase I Applications:	April 28, 2026, at 11:59 PM Eastern
Submission Deadline for FY26 Phase II Letters of Intent:	April 28, 2026, at 5 PM Eastern
Submission Deadline for FY26 Phase II Applications:	May 19, 2026, at 11:59 PM Eastern
Submission Deadline for Phase II Applications resulting from FY26 Phase I Awards:	December 17, 2026, at 11:59 PM Eastern

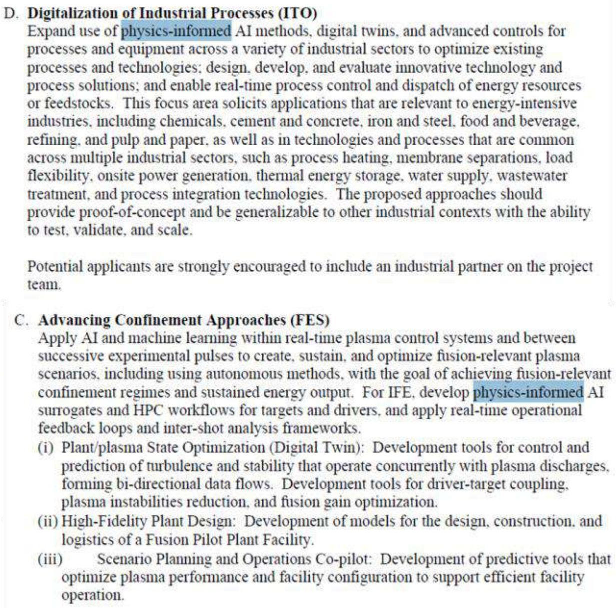


Justification: DOE's suite of world leading and unique experimental and computational capabilities for materials research, including X-ray light sources, neutron scattering facilities (and their associated characterization equipment), nanoscale science research centers, materials databases, and exascale computers, is collectively the most comprehensive and performant in the world. These capabilities, along with the availability of very large materials data sets coupled with sustained investments in the development of AI-enabled physics-informed models, has positioned DOE to take a leadership role in implementing the materials by design vision.

AI Solution: Developing AI capable of reasoning under extreme uncertainty, integrating heterogeneous data types (i.e., seismic, geochemical, biological, hydrologic), and building predictive models of systems that cannot be directly observed has broad applicability to any domain requiring inference from indirect evidence. For subsurface science, AI that connects molecular-scale mechanisms to field-scale resource availability will transform the field from descriptive to predictive. DOE's vast biological, geochemical, geomechanical, and hydrologic data sets can be combined with high performance modeling capabilities, laboratory analytics, and a suite of AI technologies, including surrogate models, physics-informed machine learning, and digital twins to enhance engineering evaluation and control of the subsurface during characterization, drilling, stimulation, and production.

AI Solution: Artificial Intelligence in Fluid Flow for Energy Components and Technologies (AFFECT) leverages sophisticated AI and machine learning techniques, including physics-informed neural networks and reinforcement learning, to analyze vast datasets from high-fidelity simulations (e.g., PDE-based computational fluid dynamics), in-situ and remote sensing observations, and flow experiments in the field and laboratory. This will enable the rapid discovery of governing equations and surrogate models for complex fluid phenomena, and optimized control strategies. The topic aims to accelerate the design cycle of energy technologies by 5-10x, reduce operational inefficiencies by 15-25%, and significantly enhance safety and reliability.

A. **Physics-Informed AI for Complex Flow Modeling (IESO, BER, ASCR, FES)**
Develops AI models that integrate fundamental physical laws and domain knowledge to accurately represent and predict complex fluid phenomena (e.g., wall-bounded turbulence, bluff-body flows, particle-fluid interactions, and atmospheric dynamics). Unlike purely data-driven models, which require vast amounts of data and may lack generalizability, physics-informed AI (e.g., PINNs, Physics-Informed Neural Networks) can learn from sparse or noisy data while adhering to known physical constraints. This improves model robustness, interpretability, and predictive capabilities across diverse operating conditions or spatial scales.



D. **Digitalization of Industrial Processes (ITO)**
Expand use of physics-informed AI methods, digital twins, and advanced controls for processes and equipment across a variety of industrial sectors to optimize existing processes and technologies; design, develop, and evaluate innovative technology and process solutions; and enable real-time process control and dispatch of energy resources or feedstocks. This focus area solicits applications that are relevant to energy-intensive industries, including chemicals, cement and concrete, iron and steel, food and beverage, refining, and pulp and paper, as well as in technologies and processes that are common across multiple industrial sectors, such as process heating, membrane separations, load flexibility, onsite power generation, thermal energy storage, water supply, wastewater treatment, and process integration technologies. The proposed approaches should provide proof-of-concept and be generalizable to other industrial contexts with the ability to test, validate, and scale.

Potential applicants are strongly encouraged to include an industrial partner on the project team.

C. **Advancing Confinement Approaches (FES)**
Apply AI and machine learning within real-time plasma control systems and between successive experimental pulses to create, sustain, and optimize fusion-relevant plasma scenarios, including using autonomous methods, with the goal of achieving fusion-relevant confinement regimes and sustained energy output. For IFE, develop physics-informed AI surrogates and HPC workflows for targets and drivers, and apply real-time operational feedback loops and inter-shot analysis frameworks.
- (i) Plant/plasma State Optimization (Digital Twin): Development tools for control and prediction of turbulence and stability that operate concurrently with plasma discharges, forming bi-directional data flows. Development tools for driver-target coupling, plasma instabilities reduction, and fusion gain optimization.
- (ii) High-Fidelity Plant Design: Development of models for the design, construction, and logistics of a Fusion Pilot Plant Facility.
- (iii) Scenario Planning and Operations Co-pilot: Development of predictive tools that optimize plasma performance and facility configuration to support efficient facility operation.

GDR CNRS IASIS: Physics-Aware Machine Learning

IASIS GT on Physics-Aware Machine Learning

https://sites.google.com/view/paml-gt-iasis/accueil

CEA-EDF-INRIA SUMMER SCHOOL PINN

Numerical Analysis school 2025: contents, program, dates, practical information.

https://ecoles-cea-edf-inria.fr/en/schools/ecole-analyse-numerique-2025/

Numerical Analysis school 2025

Solving partial differential equations in fields physics faster with physics-informed machine learning

GDR CNRS IASIS: Physics-Aware ML &Large Models

Workshop day on Physics-Informed Machine Learning and Large Models - 10 June 2025

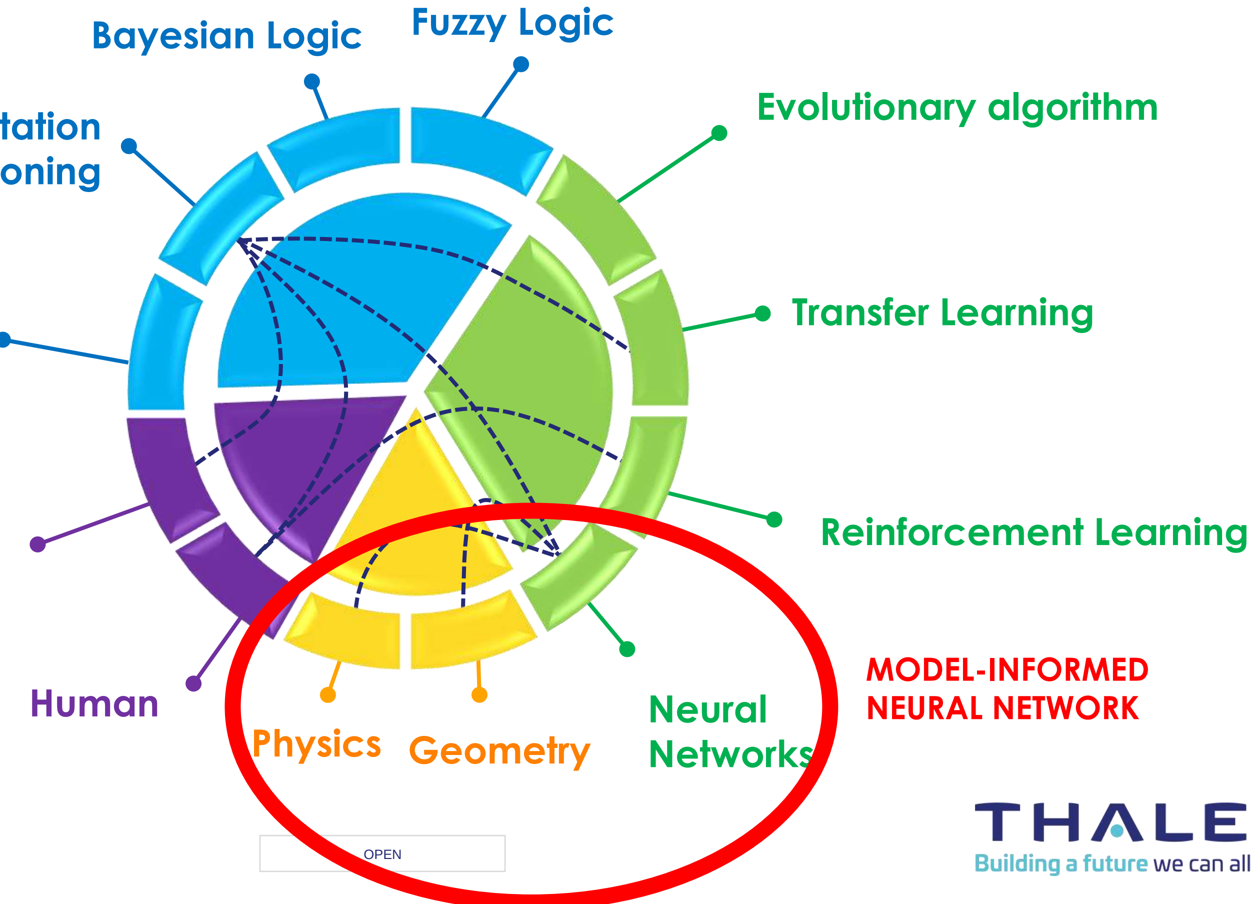
https://sites.google.com/view/paml-gt-iasis/accueil/workshop-paml-lm-june-10

CLUSTER IA ANITI: PhysimAI

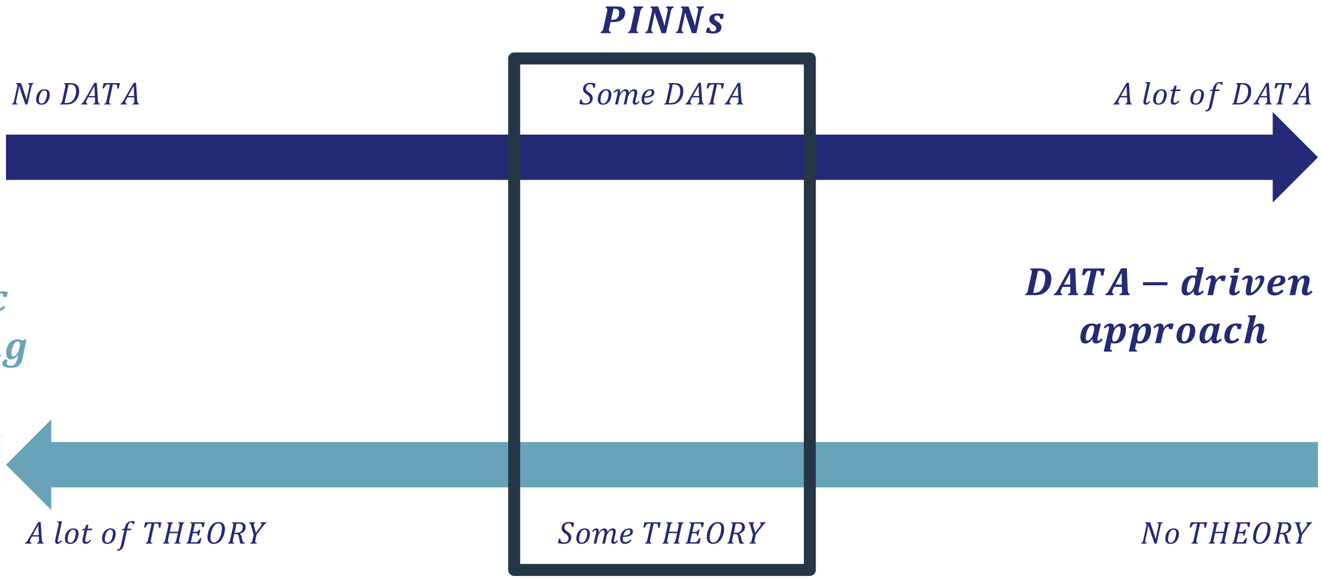
https://aniti.univ-toulouse.fr/2025/05/19/lancement-du-programme-integratif-aniti-physimai-lia-au-service-de-la-simulation-physique/

PhysimAI
for Physical Simulations
ANITI integrative Program

HYBRID AI: Model-Informed Neural Network(Geometry+Physics+Data)



Physics-Informed Neural Networks (PINNs)



PINNs

No DATA

Some DATA

A lot of DATA

Scientific computing

DATA – driven approach

A lot of THEORY

Some THEORY

No THEORY

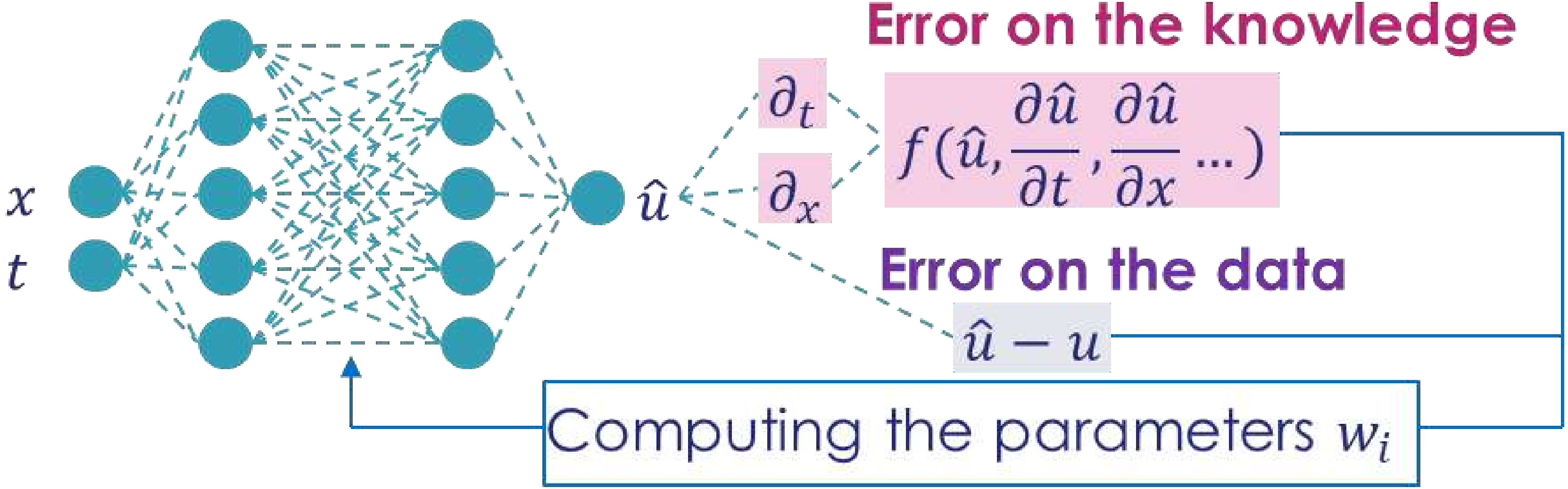
PINN: Physics-Informed Neural Network

Physical model (PDE)
- Difficult when the problem is partially observed
- Complex physics require intensive numerical methods

Deep Learning
- Lack of robustness
- Need for a lot of Data

PINNs

Analytical Model Error due to unkown phenomena is corrected by a hyhgly non-linear model (neural network)



Error on the knowledge

$$f\left(\hat{u}, \frac{\partial \hat{u}}{\partial t}, \frac{\partial \hat{u}}{\partial x} \dots\right)$$

Error on the data

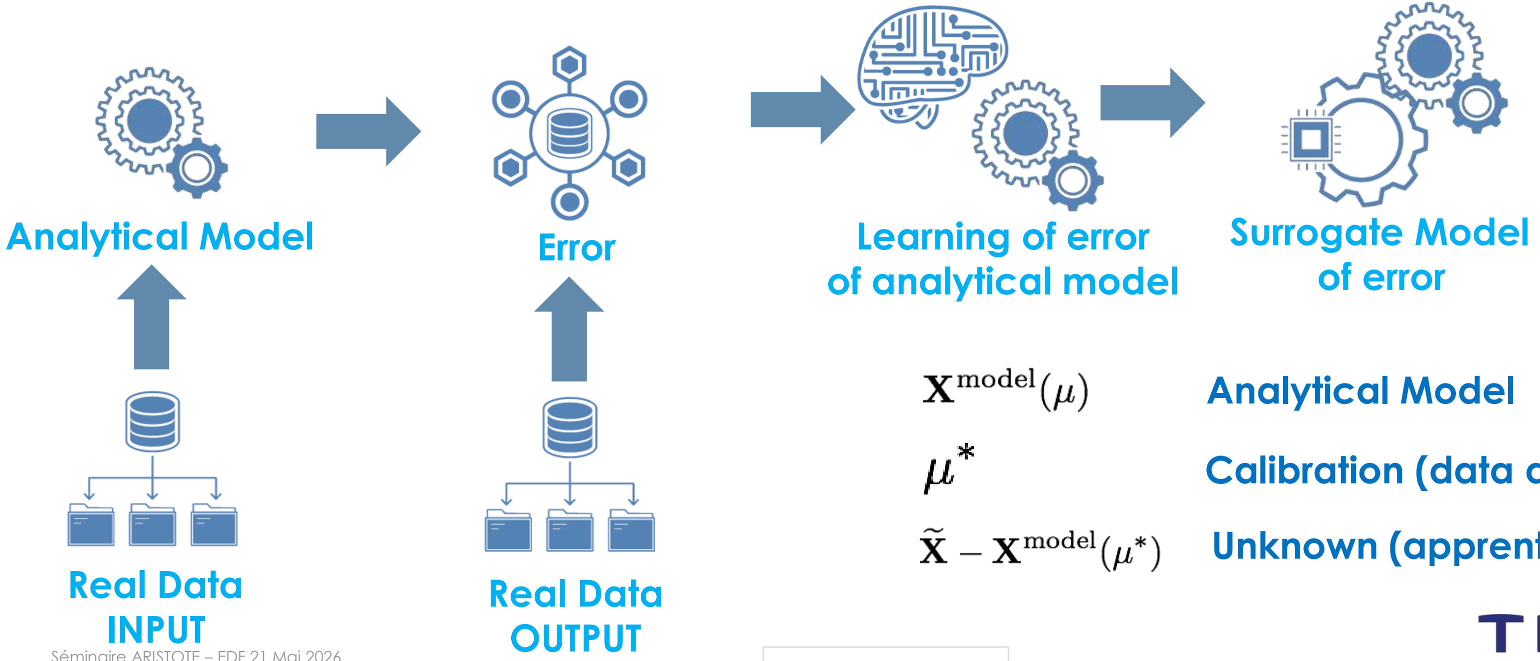
$$\hat{u} - u$$

Computing the parameters w_i

PINN (Physics-Informed Neural Network)

PINN Use-case

- **PINN Use-Case**: Hybrid Learning to learn Analytical Model Error
  - We have an analytical but imperfect model (e.g. no modeling of non-linearities or couplings). The SURROGATE model will learn the model error to refine the model.



Analytical Model

Error

Learning of error of analytical model

Surrogate Model of error

Real Data INPUT

Real Data OUTPUT

$\mathbf{X}^{\text{model}}(\mu)$	**Analytical Model**
μ^*	**Calibration (data assimilation)**
$\widetilde{\mathbf{X}} - \mathbf{X}^{\text{model}}(\mu^*)$	**Unknown (apprentissage)**

Advanced Analytical Model-Informed Machine Learning

Symplectic-Informed Neural Network

An accurate data-based prediction of the long-term evolution of Hamiltonian systems requires a network that preserves the appropriate structure under each time step. Every Hamiltonian system contains two essential ingredients: the Poisson bracket and the Hamiltonian. Hamiltonian systems with symmetries, whose paradigm examples are the Lie–Poisson systems, have been shown to describe a broad category of physical phenomena, from satellite motion to underwater vehicles, fluids, geophysical applications, complex fluids, and plasma physics.

:Algorithms

Lie-Poisson Neural Network
(LPNet: F. Gay-Balmaz)

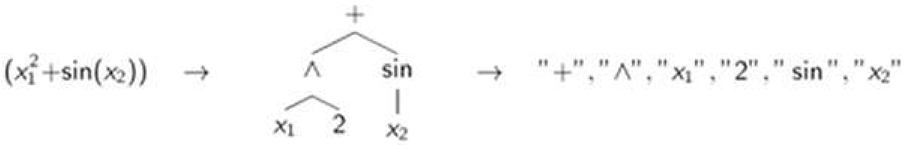
$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}$$

$$\frac{dF}{dt} = \{F, H\} = \frac{\partial F}{\partial \mathbf{q}}\frac{\partial H}{\partial \mathbf{p}} - \frac{\partial H}{\partial \mathbf{q}}\frac{\partial F}{\partial \mathbf{p}}$$

Language Model Informed
(AI Scientist Tools)

Is it possible to teach complicated mathematical problems to an AI that has no mathematical knowledge? Using language models to give intuitions on solving open mathematical problems.

Language Model-Informed to Solve Math
(LMI: Amaury Hayat)



SINN & MINN: Symplectic-Informed NN & Metriplectic-Informed NN

Symplectic Integrator

- Capture of symmetries (Geometry-Informed)
- Capture Noether Invariants via Souriau's Moment Map as Energy, Angular Momentum (Physics-Informed)

$$\frac{dF}{dt} = \{F, H\}$$

non-dissipative

(*) The time evolution of Hamilton's equations

$$\dot{p} = -\frac{\partial H}{\partial q} \quad , \quad \dot{q} = \frac{\partial H}{\partial p}$$

is a symplectomorphism, meaning that it conserves the symplectic 2-form $dp \wedge dq$. A numerical scheme is a symplectic integrator if it also conserves this 2-form and equation are reduce to:

$$z = (p, q) \Rightarrow \dot{z} = \{z, H(z)\}$$

Metriplectic Integrator

- Capture 1st principle (Energy preservation, Moment map)
- Capture 2nd principle (Entropy Production)

$$\frac{dF}{dt} = \{F, H\} + (F, S)$$

dissipative

$$F = H + S$$

(*) 1st principle: Preservation of Energy

$$\frac{dH}{dt} = 0$$

2nd Principle Entropy production

$$\frac{dS}{dt} \geq 0$$

Thermodynamics-Informed Neural Network

$$\frac{dz}{dt} = L(z)\frac{\partial E}{\partial z} + M(z)\frac{\partial S}{\partial z}$$

$$\frac{dz}{dt} = \{z, E\} + (z, S)$$

Metriplectic Flow

- This same assumption, $k_B \to 0$, induces two additional consequences:

$$L(z)\frac{\partial S}{\partial z} = 0 \quad \text{and} \quad M(z)\frac{\partial E}{\partial z} = 0$$

- which constitute the ingredients of the celebrated General Equation for the non-Equilibrium Reversible-Irreversible Coupling, **GENERIC equations**. This type of formulations are also known as metriplectic formulations, since they combine metric and symplectic terms. However, in GENERIC equations, known as degeneracy conditions, play a fundamental role. They are key ingredients in the demonstration of the a priori satisfaction of the two laws of thermodynamics:
  - **Conservation of energy** in closed systems Given the **anti-symmetry of L**:

$$\frac{dE}{dt}(z) = \frac{\partial E}{\partial z}\frac{\partial z}{\partial t} = 0$$

  - **Non-negative entropy production**, given the **positive semi-definiteness of M**:

$$\frac{dS}{dt}(z) = \frac{\partial S}{\partial z}\frac{\partial z}{\partial t} = \frac{\partial S}{\partial z} M(z) \frac{\partial S}{\partial z} \geq 0$$

Thermodynamics-Informed Neural Network

- Therefore, the GENERIC structure consistently guarantees the satisfaction of the laws of thermodynamics by construction. This makes GENERIC a very appealing choice for the construction of inductive biases in the learning of physical phenomena.
- For Thermodynamics of learning physical phenomena, we assume that D_i data sets contain labelled pairs of a single-step state vector z_t and its time evolution z_{t+1}

$$D=\left\{D_i\right\}_{i=1}^{N_{sim}},D_i=\left\{\left(z_t,z_{t+1}\right)\right\}_{t=0}^{T}$$

- so that a neural network can be constructed by means of two loss terms, a data loss term that takes into account the correct prediction of the state vector time evolution using the GENERIC integrator, defined as:

$$\Lambda_n^{data}=\left\|\frac{dz^{GT}}{dt}-\frac{dz^{net}}{dt}\right\|_2^2$$

Thermodynamics-Informed Neural Network

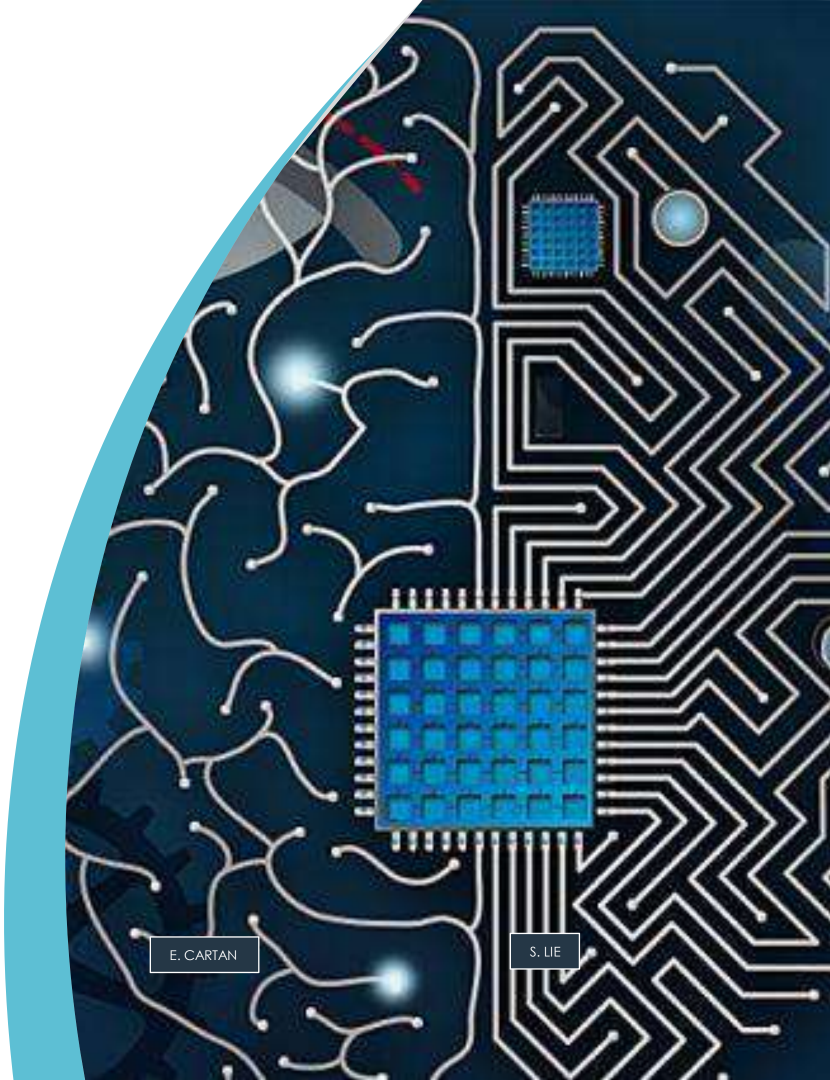
- where $\|\cdot\|_2$ denotes the L2-norm. The choice of the time derivative instead of the state vector itself is to regularize the global loss function to a uniform order of magnitude with respect to the degeneracy terms. A second loss term takes into account the fulfillment of the degeneracy equations,

$$\Lambda_n^{\deg} = \left\| L \frac{\partial S}{\partial z_n} \right\|_2^2 + \left\| M \frac{\partial E}{\partial z_n} \right\|_2^2$$

- This formulation gave rise to the so-called structure-preserving neural networks and thermodynamics-informed neural networks. These networks have been employed recently in the development of physics perception with the help of computer vision techniques.
- The global loss term is a weighted mean of the two terms over the shuffled Nbatch batched snapshots.

$$\Lambda = \frac{1}{N_{batch}} \sum_{n=0}^{N_{batch}} \left(\lambda \Lambda_n^{data} + \Lambda_n^{\deg} \right)$$

From Transverse Symplectic Foliation to Dissipative PINN



Jean-Marie Souriau (1922-2012): New way of thinking Physics

« Il est évident que l'on ne peut définir de valeurs moyennes que sur des objets appartenant à un espace vectoriel (ou affine); donc - si bourbakiste que puisse sembler cette affirmation - que l'on n'observera et ne mesurera de valeurs moyennes que sur des grandeurs appartenant à un ensemble possédant physiquement une structure affine. Il est clair que cette structure est nécessairement unique - sinon les valeurs moyennes ne seraient pas bien définies. » -

Jean-Marie Souriau

« Il n'y a rien de plus dans les théories physiques que les groupes de symétrie si ce n'est la construction mathématique qui permet précisément de montrer qu'il n'y a rien de plus » -

Jean-Marie Souriau

ENTROPY DEFINITION: Entropy is Invariant Casimir Function on Leaves of Symplectic Foliation associated to Coadjoint Orbits generated via Moment Map of Symmetry Group acting on the System.

Claude Shannon

$$\{S,H\}_{\tilde{\Theta}}(Q)=0$$

$$ad^{*}_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0$$

$$\beta\in Ker\tilde{\Theta}_{\beta}\Rightarrow\left\langle Q,\left[\frac{\partial S}{\partial Q},Z\right]\right\rangle+\tilde{\Theta}\left(\frac{\partial S}{\partial Q},Z\right)=0$$

$$\langle Q,[\beta,Z]\rangle+\tilde{\Theta}(\beta,Z)=0 \text{ with } \beta=\frac{\partial S}{\partial Q}$$

β : Planck Temperature ; Q : Heat

Jean-Marie Souriau (ENS'42)

THALES
Building a future we can all trust

Mécanique Statistique, Groupes de Lie et Cosmologie:
Formulation Symplectique de la mécanique statistique

Colloques Internationaux C.N.R.S.
N° 237 – Géométrie symplectique et physique mathématique

MÉCANIQUE STATISTIQUE,
GROUPES DE LIE ET COSMOLOGIE

Jean-Marie SOURIAU (1)

Première partie
FORMULATION SYMPLECTIQUE DE LA MECANIQUE STATISTIQUE

RESUME

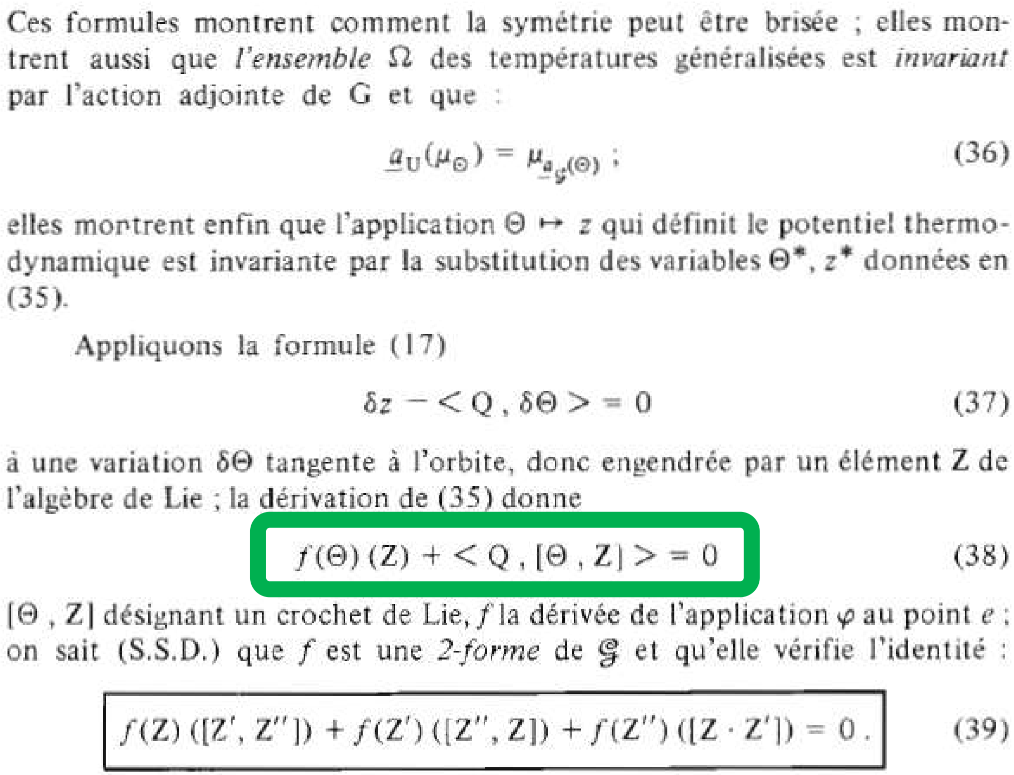
La notion classique d'ensemble canonique de Gibbs est étendue au cas d'une variété symplectique sur laquelle un groupe de Lie possède une action symplectique ("groupe dynamique").

La définition rigoureuse donnée ici permet d'étendre un certain nombre de propriétés thermodynamiques classiques (la température est ici un élément de l'algèbre de Lie du groupe, la chaleur un élément de son dual), notamment des inégalités de convexité.

Dans le cas de groupes non commutatifs, des propriétés particulières apparaissent : la symétrie est spontanément brisée, certaines relations de type cohomologique sont vérifiées dans l'algèbre de Lie du groupe.

Diverses applications sont abordées (corps tournants, mécanique statistique covariante ou relativiste).

[Ces résultats précisent et complètent une étude publiée dans un ouvrage antérieur (²) qui sera désigné par les initiales S.S.D.].



Ces formules montrent comment la symétrie peut être brisée ; elles montrent aussi que *l'ensemble* Ω des températures généralisées est *invariant* par l'action adjointe de G et que :

$$\underline{a}_U(\mu_\Theta) = \mu_{\underline{a}_{\mathfrak{g}}(\Theta)} \; ; \qquad (36)$$

elles montrent enfin que l'application $\Theta \mapsto z$ qui définit le potentiel thermodynamique est invariante par la substitution des variables Θ^*, z^* données en (35).

Appliquons la formule (17)

$$\delta z - \langle Q \, , \delta\Theta \rangle = 0 \qquad (37)$$

à une variation $\delta\Theta$ tangente à l'orbite, donc engendrée par un élément Z de l'algèbre de Lie ; la dérivation de (35) donne

$$\boxed{f(\Theta)(Z) + \langle Q \, , [\Theta \, , Z] \rangle = 0} \qquad (38)$$

$[\Theta \, , Z]$ désignant un crochet de Lie, f la dérivée de l'application φ au point e ; on sait (S.S.D.) que f est une *2-forme* de $\mathfrak{G}$ et qu'elle vérifie l'identité :

$$\boxed{f(Z)([Z', Z'']) + f(Z')([Z'', Z]) + f(Z'')([Z \cdot Z']) = 0 \, .} \qquad (39)$$

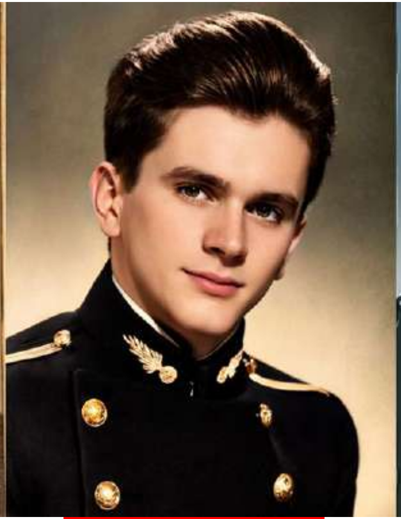
Séminaire Sud-Rhodanien

Le Séminaire "Sud-Rhodanien de Géométrie": esprits raffinés et ingénieurs savants, qui sur les fondations de leurs ainés, Elie Cartan, André Lichnerowicz et Charles Ehresmann, ont élaboré l'architecture des structures de la géométrie symplectique, des feuilletages et de la quantification géométrique pour la mécanique hamiltonienne et la mécanique quantique. Créé au début des années 1980 par les universités d'Avignon, Aix-Marseille, Lyon et Montpellier, ce séminaire contribua de manière importante au développement des recherches faites en France sur la géométrie symplectique et la géométrie de Poisson.

Livres du séminaire Sud-Rhodanien de Géométrie
Volume 1. Géométrie symplectique et de contact
Volume 2. Feuilletages et quantification géométrique
Volume 3. Autour du théorème de Poincaré-Birkhoff
Volume 4. Géométrie symplectique et mécanique
Volume 5. Singularités, feuilletages et mécanique hamiltonienne
Volume 6. Aspects dynamiques et topologiques des groupes infinis de transformation de la mécanique
Volume 7. Feuilletages riemanniens, quantification géométrique et mécanique
Volume 8. Actions hamiltoniennes de groupe. Troisième théorème de Lie
Volume 9. Symplectic Geometry, Groupoids, and Integrable Systems - à Berkeley


JEAN-MARIE SOURIAU


PIERRE DAZORD


PIERRE MOLINO


CHARLES-MICHEL MARLE


PAULETTE LIBERMANN

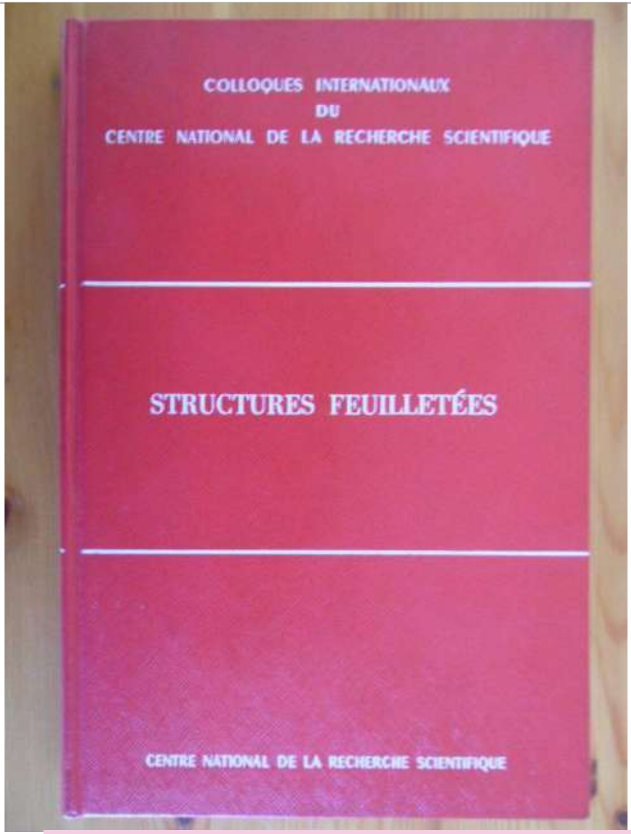
Femal Touch of Symplectic Geometry

5 Masterminds in Symplectic Geometry

- **Paulette LIBERMANN**
- **Michèle VERGNE**
- **Michèle AUDIN**
- **Yvette KOSMANN-SCHWARZBACH**

1963, Birth of « Foliated structures »

C. EHRESMANN, Structures feuilletées, Proceedings of the Fifth Canadian Mathematical Congress, 109-172., 1963


COLLOQUES INTERNATIONAUX
DU
CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

STRUCTURES FEUILLETÉES

CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

LISTE DES INVITÉS OFFICIELS DU COLLOQUE

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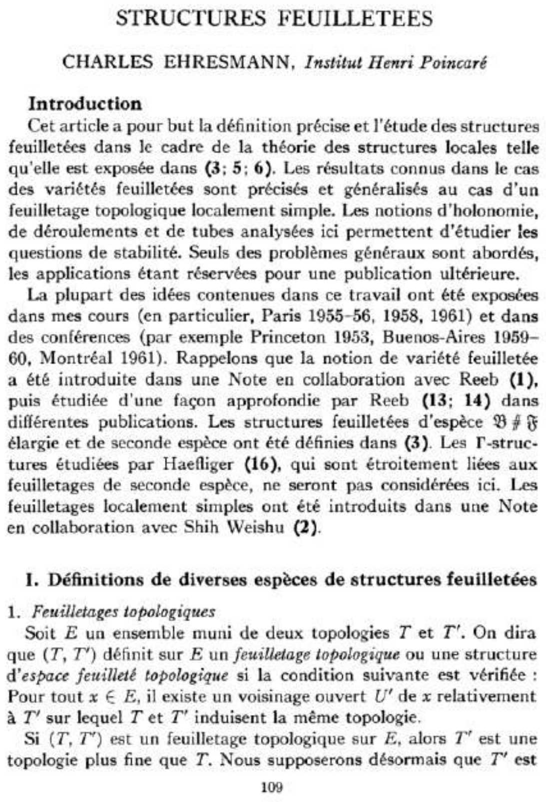
L. **Markus** (Minneapolis),
H. **Osborn** (Illinois),
M. **Peixoto** (Rio-de-Janeiro),



STRUCTURES FEUILLETEES

CHARLES EHRESMANN, *Institut Henri Poincaré*

Introduction

Cet article a pour but la définition précise et l'étude des structures feuilletées dans le cadre de la théorie des structures locales telle qu'elle est exposée dans **(3; 5; 6)**. Les résultats connus dans le cas des variétés feuilletées sont précisés et généralisés au cas d'un feuilletage topologique localement simple. Les notions d'holonomie, de déroulements et de tubes analysées ici permettent d'étudier les questions de stabilité. Seuls des problèmes généraux sont abordés, les applications étant réservées pour une publication ultérieure.

La plupart des idées contenues dans ce travail ont été exposées dans mes cours (en particulier, Paris 1955–56, 1958, 1961) et dans des conférences (par exemple Princeton 1953, Buenos-Aires 1959–60, Montréal 1961). Rappelons que la notion de variété feuilletée a été introduite dans une Note en collaboration avec Reeb **(1)**, puis étudiée d'une façon approfondie par Reeb **(13; 14)** dans différentes publications. Les structures feuilletées d'espèce $\mathfrak{B} \# \mathfrak{F}$ élargie et de seconde espèce ont été définies dans **(3)**. Les Γ-structures étudiées par Haefliger **(16)**, qui sont étroitement liées aux feuilletages de seconde espèce, ne seront pas considérées ici. Les feuilletages localement simples ont été introduits dans une Note en collaboration avec Shih Weishu **(2)**.

I. Définitions de diverses espèces de structures feuilletées

1. *Feuilletages topologiques*

Soit E un ensemble muni de deux topologies T et T'. On dira que (T, T') définit sur E un *feuilletage topologique* ou une structure d'*espace feuilleté topologique* si la condition suivante est vérifiée : Pour tout $x \in E$, il existe un voisinage ouvert U' de x relativement à T' sur lequel T et T' induisent la même topologie.

Si (T, T') est un feuilletage topologique sur E, alors T' est une topologie plus fine que T. Nous supposerons désormais que T' est

109

Structures feuilletées - Grenoble, 25-30 juillet 1963 - Georges Reeb
Auteur(s) : Georges Reeb - Charles Ehresmann - René Thom - Paulette Libermann
Editeur : Centre National De La Recherche ScientifiqueCollection : Colloques Int. Du Cnrs, 1964

Transverse Structure of Foliations

116 ASTÉRISQUE

1984

STRUCTURE TRANSVERSE DES FEUILLETAGES

Toulouse, 17-19 février 1982

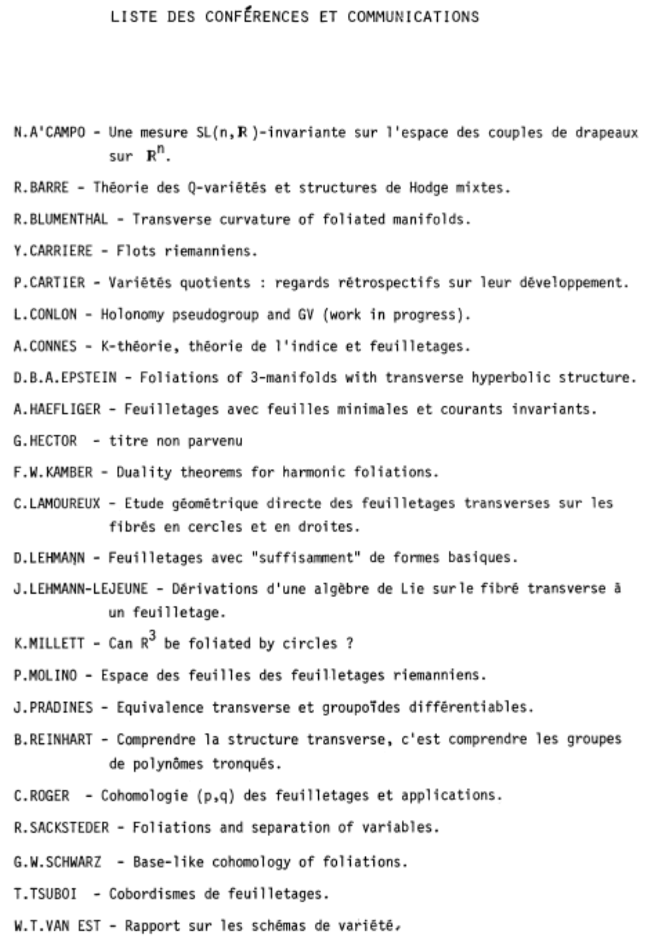
SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Astérisque | no. 116

Structure transverse des feuilletages

Collectif

Astérisque, no. 116 (1984) , 302 p.



LISTE DES CONFÉRENCES ET COMMUNICATIONS

N.A'CAMPO - Une mesure $SL(n,\mathbb{R})$-invariante sur l'espace des couples de drapeaux sur $\mathbb{R}^n$.

R.BARRE - Théorie des Q-variétés et structures de Hodge mixtes.

R.BLUMENTHAL - Transverse curvature of foliated manifolds.

Y.CARRIERE - Flots riemanniens.

P.CARTIER - Variétés quotients : regards rétrospectifs sur leur développement.

L.CONLON - Holonomy pseudogroup and GV (work in progress).

A.CONNES - K-théorie, théorie de l'indice et feuilletages.

D.B.A.EPSTEIN - Foliations of 3-manifolds with transverse hyperbolic structure.

A.HAEFLIGER - Feuilletages avec feuilles minimales et courants invariants.

G.HECTOR - titre non parvenu

F.W.KAMBER - Duality theorems for harmonic foliations.

C.LAMOUREUX - Etude géométrique directe des feuilletages transverses sur les fibrés en cercles et en droites.

D.LEHMANN - Feuilletages avec "suffisamment" de formes basiques.

J.LEHMANN-LEJEUNE - Dérivations d'une algèbre de Lie sur le fibré transverse à un feuilletage.

K.MILLETT - Can R^3 be foliated by circles ?

P.MOLINO - Espace des feuilles des feuilletages riemanniens.

J.PRADINES - Equivalence transverse et groupoïdes différentiables.

B.REINHART - Comprendre la structure transverse, c'est comprendre les groupes de polynômes tronqués.

C.ROGER - Cohomologie (p,q) des feuilletages et applications.

R.SACKSTEDER - Foliations and separation of variables.

G.W.SCHWARZ - Base-like cohomology of foliations.

T.TSUBOI - Cobordismes de feuilletages.

W.T.VAN EST - Rapport sur les schémas de variété.

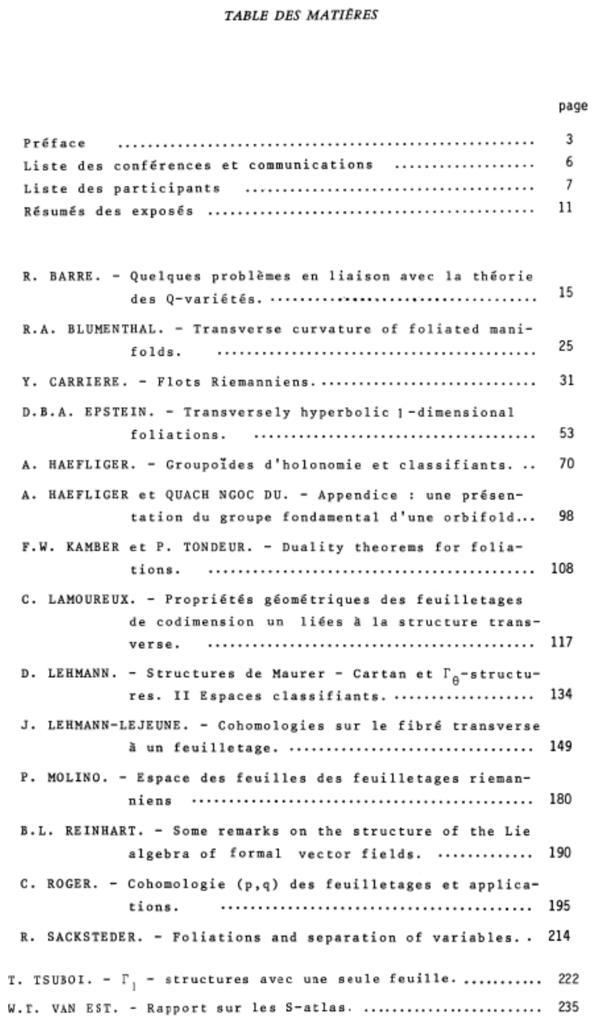


TABLE DES MATIÈRES

http://www.numdam.org/item/AST_1984__116_/

European Actions: COST CaLISTA, MSCA CaLIGOLA, MaGyQ

Cartan geometry, Lie and representation theory, Integrable Systems, quantum Groups and quantum computing towards the understanding of the geometry of deep Learning and its Applications

CaLISTA

CaLIGOLA

https://site.unibo.it/calista/en

https://site.unibo.it/caligola/en

Elie CARTAN
(Student of Ecole Normale Supérieure Paris)

Funded by the European Union

CARTAN GEOMETRY

MaGyQ

https://sites.google.com/view/magyqproject

AI/Machine Learning Evolution: ALGEBRA COMPUTATION STRUCTURES

Calcul formel pour les méthodes de Lie en mécanique hamiltonienne
P.V. Koseleff, X/CMLS PhD, 1993 (P. Cartier)

Souriau Exponential Map Algorithm for Machine Learning on Matrix Lie Groups
Frédéric Barbaresco, Springer GSI'19, 2019

Supervarieties, Sow. Math. Dokl. 16 (1975), 1218-1222.
F. A. Berzin and D. A. Leites

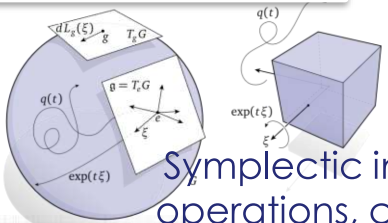
LIE SUPER ALGEBRA

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$\mathrm{Ber}(X) = \det(A)\det(D - CA^{-1}B)^{-1}$$

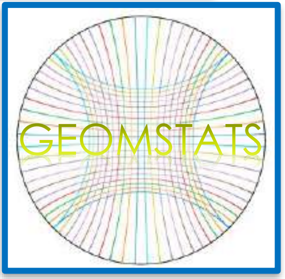
Berezian Determinant

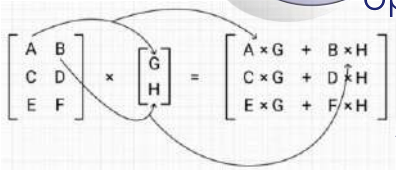
LIE ALGEBRA



Symplectic integrators, non-commutative operations, coadjoint orbits, moment map

Computer Algebra Group- Scratchpad, IBM,1971





LINEAR ALGEBRA

Vectors space, commutative matrix operations, eigen-analysis

GOOGLE TPU (Tensor Processing Unit)

x	y	x AND y	x OR y	NOT x	NOT y
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	1
1	1	1	1	0	0

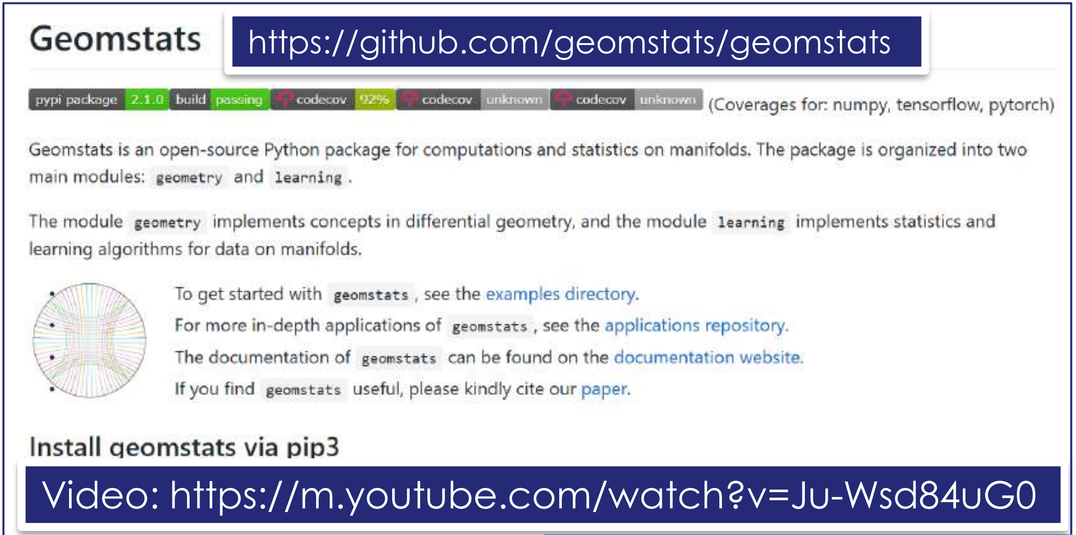
BOOLE ALGEBRA

Boolean logic digital circuits using electromechanical relays as the switching element.

George R. Stibitz (Bell Labs)

ALGEBRA is the study of mathematical symbols and the rules for manipulating these symbols

GEOMSTATS: PYTHON Library for Lie Group Machine Learning



Geomstats

https://github.com/geomstats/geomstats

pypi package 2.1.0 build passing codecov 92% codecov unknown codecov unknown (Coverages for: numpy, tensorflow, pytorch)

Geomstats is an open-source Python package for computations and statistics on manifolds. The package is organized into two main modules: `geometry` and `learning`.

The module `geometry` implements concepts in differential geometry, and the module `learning` implements statistics and learning algorithms for data on manifolds.

To get started with `geomstats`, see the examples directory.

For more in-depth applications of `geomstats`, see the applications repository.

The documentation of `geomstats` can be found on the documentation website.

If you find `geomstats` useful, please kindly cite our paper.

Install geomstats via pip3

Video: https://m.youtube.com/watch?v=Ju-Wsd84uG0

```
pip3 install geomstats
```

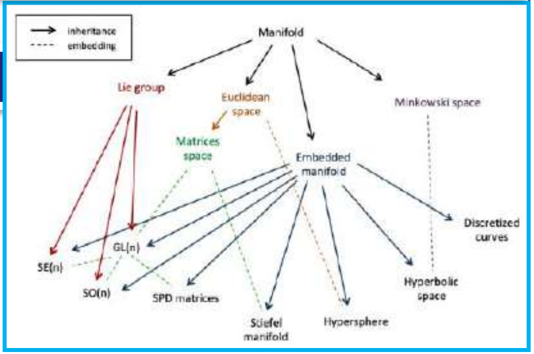
hal-02536154, version 1

Pré-publication, Document de travail

https://hal.inria.fr/hal-02536154

Geomstats: A Python Package for Riemannian Geometry in Machine Learning

Nina Miolane [1], Alice Le Brigant, Johan Mathe [2], Benjamin Hou [3], Nicolas Guigui [4, 5], Yann Thanwerdas [4, 5], Stefan Heyder [6], Olivier Peltre, Niklas Koep, Hadi Zaatiti [7], Hatem Hajri [7], Yann Cabanes, Thomas Gerald, Paul Chauchat [8], Christian Shewmake, Bernhard Kainz, Claire Donnat [9], Susan Holmes [1], Xavier Pennec [4, 5] Détails



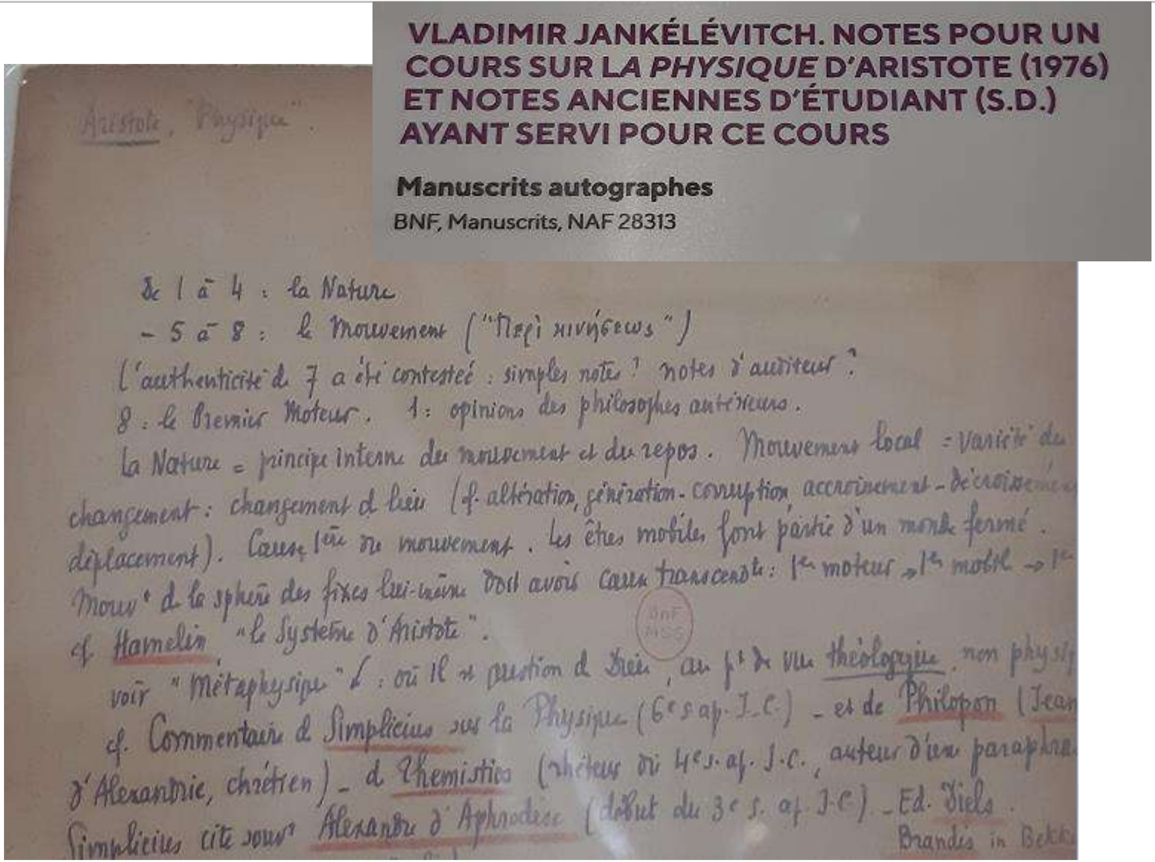
Les quatre catégories du mouvement dans LA PHYSIQUE d'ARISTOTE: Les livres V et VI traitent des modalités du mouvement. Il en existe quatre catégories : **un mouvement de quantité, de qualité, de lieu et de substance**. Le mouvement de substance est une génération ou corruption.

Aristote & Pierre Duhem: Mouvement = changement de lieu (mécanique) et changement d'état (la thermodynamique)

P. DUHEM

ARISTOTLE

Aristotle's PHYSICS: Movment as change of place & change of quality



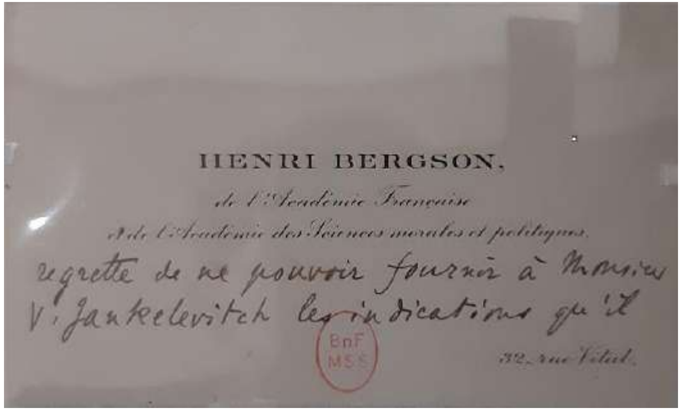
VLADIMIR JANKÉLÉVITCH. NOTES POUR UN COURS SUR LA *PHYSIQUE* D'ARISTOTE (1976) ET NOTES ANCIENNES D'ÉTUDIANT (S.D.) AYANT SERVI POUR CE COURS

Manuscrits autographes

BNF, Manuscrits, NAF 28313

Aristote, "Physique".

Lc 1 à 4 : la Nature
– 5 à 8 : le Mouvement ("Περὶ κινήσεως")
(l'authenticité de 7 a été contestée : simples notes ? notes d'auditeur ?
8 : le Premier Moteur. 1 : opinions des philosophes antérieurs.
La Nature = principe interne du mouvement et du repos. Mouvement local = variété du changement : changement de lieu (cf. altération, génération-corruption, accroissement-décroisseme déplacement). Cause 1ère du mouvement. Les êtres mobiles font partie d'un monde fermé. Mouvt de la sphère des fixes lui-même doit avoir cause transcendte : 1er moteur → 1er mobile → 1er
cf. Hamelin "le Système d'Aristote".
voir "Métaphysique" Λ : où il est question de Dieu, au pt de vue théologique, non physiq
cf. Commentaire de Simplicius sur la Physique (6e s. ap. J.-C.) – et de Philopon (Jean d'Alexandrie, chrétien) – de Themistius (rhéteur du 4e s. ap. J.-C., auteur d'une paraphra
Simplicius cite souvent Alexandre d'Aphrodise (début du 3e s. ap. J.-C.). – Ed. Diels.
Brandis in Bekk



HENRI BERGSON,
de l'Académie Française
et de l'Académie des Sciences morales et politiques,
regrette de ne pouvoir fournir à Monsieur V. Jankelevitch les indications qu'il
32, rue Vital.

Bibliothèque François-Mitterrand – Exposition Vladimir Jankelevitch - 2019

La philosophie naturelle d'Aristote et l'épistémologie de Pascal

- “***Nous avons fait de la Dynamique un cas particulier de la Thermodynamique, une Science qui embrasse dans des principes communs tous les changements d’état des corps, aussi bien les changements de lieu que les changements de qualités physiques***” - Pierre Duhem, Sur les équations générales de la Thermodynamique, 1891
- “***Nous prenons le mot mouvement pour désigner non seulement un changement de position dans l’espace, mais encore un changement d’état quelconque, lors même qu’il ne serait accompagné d’aucun déplacement . . . De la sorte, le mot mouvement s’oppose non pas au mot repos, mais au mot équilibre.***” - Pierre Duhem, Commentaire aux principes de la Thermodynamique, 1894
- “***This theoretical design led Duhem to rediscover and reinterpret the tradition of Aristotle’s natural philosophy and Pascal’s epistemology . . . This outcome was surprising and clearly echoed the Aristotelian language and concept of motion as change and transformation: within the framework of Aristotelian natural philosophy, motion in the modern physical sense was actually a special case of the general concept of motion… Pascal had criticised Descartes’s mechanism and had stressed the need for a theoretical practice, a synthetic and intuitive one, besides the geometric formalisation***” – S. Bordoni, From thermodynamics to philosophical tradition: Pierre Duhem’s research between 1891 and 1896. Lettera Matematica 2017, 5, 261–266.

Mouvement et philosophie naturelle d’Aristote

Duhem, P. (1894), Commentaire aux principes de la Thermodynamique – Troisième partie, Journal de Mathématiques pures et appliquées, 4e série, tome 10, 207-285.

- *“Il nous semble qu’une conclusion générale se dégage de cette étude: si la science des mouvements cesse d’être, dans l’ordre logique, la première des Sciences physiques, pour devenir seulement un cas particulier* ***d’une science plus générale embrassant dans ses formules toutes les modifications des corps****, la tentation sera moindre, pensons-nous, de ramener l’étude de tous les phénomènes physiques à l’étude du mouvement; on comprendra mieux que* ***le changement de lieu dans l’espace n’est pas une modification plus simple que le changement de température ou de quelque autre qualité physique****; on fuira dès lors plus volontiers ce qui a été jusqu’ici le plus dangereux écueil de la Physique théorique, la recherche d’une explication mécanique de l’Univers.”*

STANLEY L. JAKI

UNEASY GENIUS:
THE LIFE AND WORK OF
PIERRE DUHEM

At fourteen

At seventeen

At nineteen

Pierre Duhem: La théorie Physique: son objet, sa structure

Pierre Duhem

- *« Il est une Cosmologie avec laquelle la Thermodynamique générale présente une analogie non-méconnaissable; cette Cosmologie,* ***c'est la Physique péripatéticienne*** *… Parmi les attributs de la substance, la Physique péripatéticienne confère une* ***égale importance à la catégorie de la quantité et à la catégorie de la qualité****; or, par ses symboles numériques, la Thermodynamique générale représente également les diverses grandeurs des quantités et les diverses intensités des qualités.* ***Le mouvement local n'est, pour Aristote, qu'une des formes du mouvement général****, tandis que les Cosmologies cartésienne, atomistique et newtonienne concordent en ceci que le seul mouvement possible est le changement de lieu dans l'espace. Et voici que la Thermodynamique générale traite, en ses formules, d'une* ***foule de modifications telles que les variations de températures, les changements d'état électrique ou d'aimantation, sans chercher le moins du monde à réduire ces variations au mouvement local »***

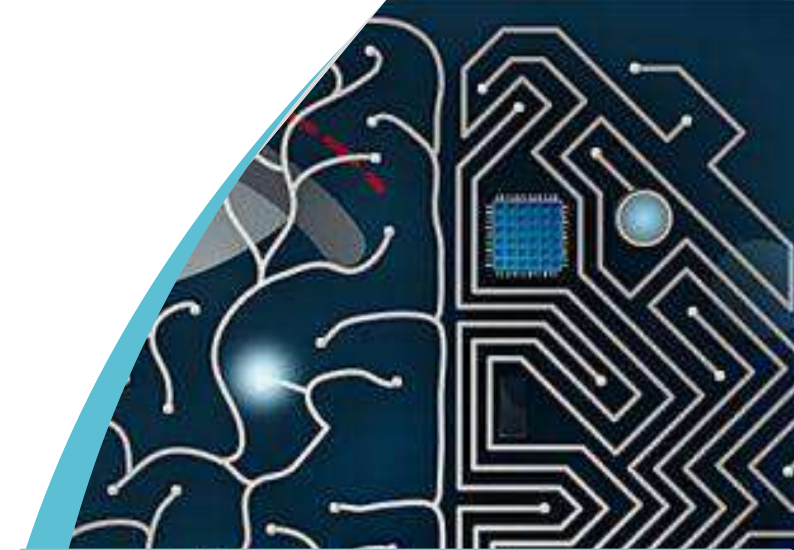
Talk outline

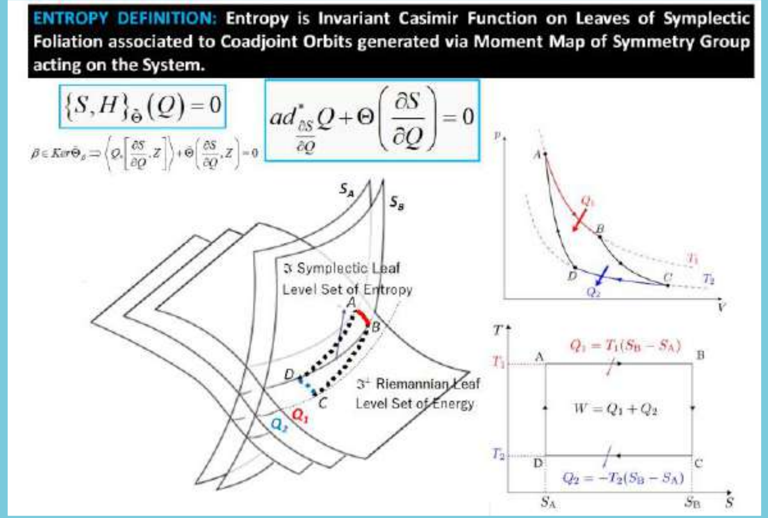
« There is nothing more in physical theories than symmetry groups except the mathematical construction which allows precisely to show that there is nothing more » -
Jean-Marie Souriau

1. Elementary Symplectic Structures of Thermodynamics
2. Constantin Carathéodory: From the Pfaffian forms to Thermodynamics Axiomatization, & Adiabatic Foliation by Frankel
3. Information Geometry & Natural Gradient for Learning
4. Jean-Marie Souriau: Symplectic Foliation via Moment Map and Koszul-Fisher Metric from Information Geometry for Lie Groups Thermodynamics
5. ENTROPY as Casimir Function in Coadjoint Representation Constant on Symplectic Leaves & Density of Maximum Entropy on Lie Groups
6. Lars Onsager & Baptiste Coquinot: Symplectic Foliation Model of METRIPLECTIC FLOW
7. Transverse Symplectic Foliation Structure
8. Transitive PINN for Drone Flight Mechanics with Friction & Drag Forces
9. Conclusion

THALES

Elementary Symplectic Structures of Thermodynamics





ENTROPY DEFINITION: Entropy is Invariant Casimir Function on Leaves of Symplectic Foliation associated to Coadjoint Orbits generated via Moment Map of Symmetry Group acting on the System.

$$\boxed{\{S,H\}_{\tilde{\Theta}}(Q)=0}$$

$$\beta\in Ker\tilde{\Theta}_{\beta}\Rightarrow\left\langle Q,\left[\frac{\partial S}{\partial Q},Z\right]\right\rangle+\tilde{\Theta}\left(\frac{\partial S}{\partial Q},Z\right)=0$$

$$\boxed{ad^{*}_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0}$$

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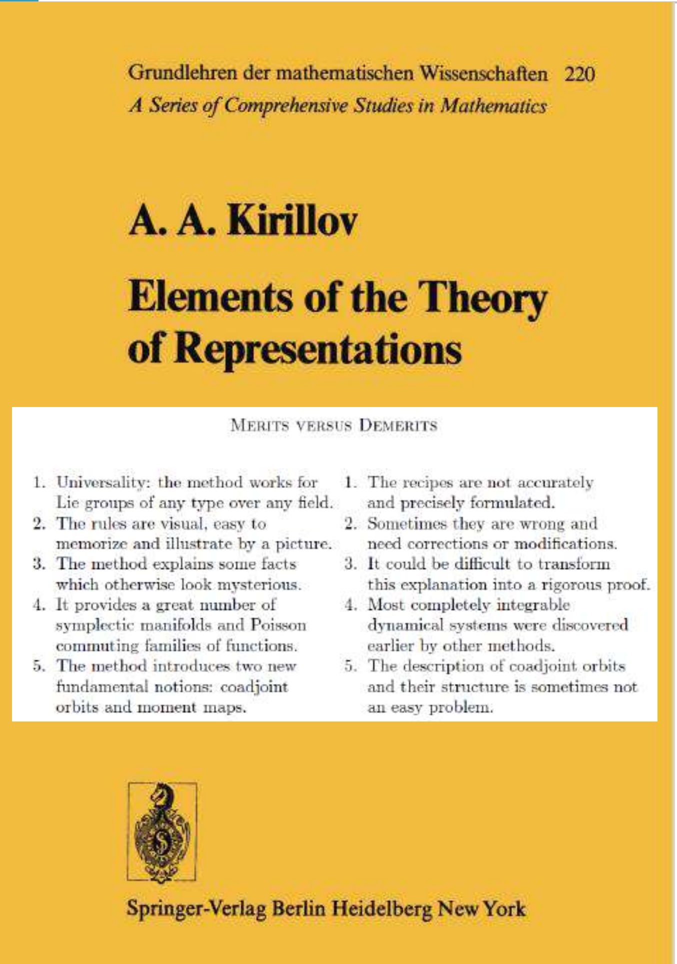
Representation Theory & (Co-adjoint) Orbits Method: A. Kirillov



Grundlehren der mathematischen Wissenschaften 220
A Series of Comprehensive Studies in Mathematics

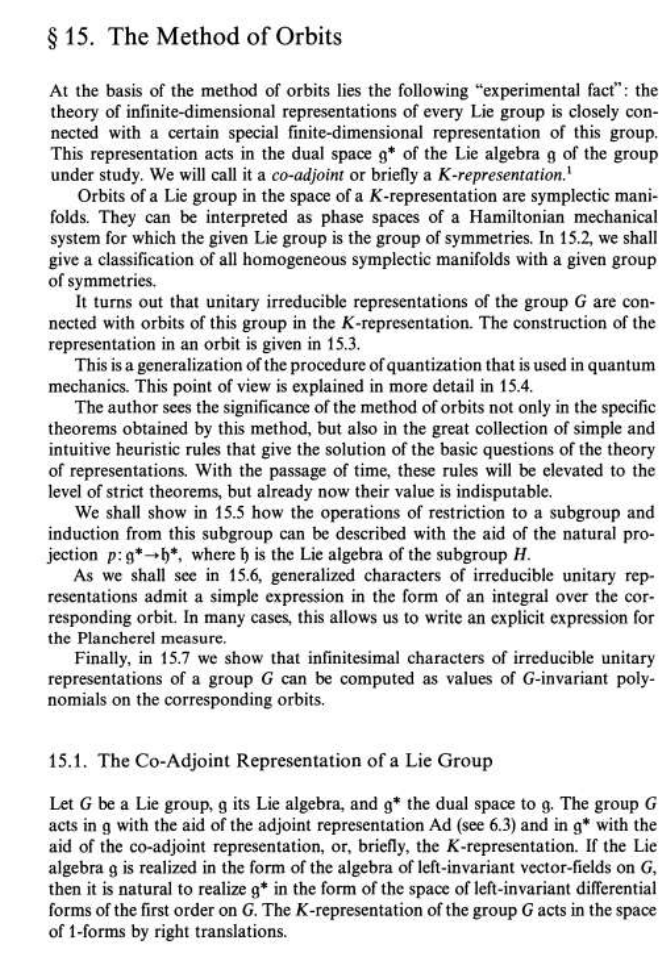
A. A. Kirillov

Elements of the Theory of Representations

Springer-Verlag Berlin Heidelberg New York

MERITS VERSUS DEMERITS

1. Universality: the method works for Lie groups of any type over any field.	1. The recipes are not accurately and precisely formulated.
2. The rules are visual, easy to memorize and illustrate by a picture.	2. Sometimes they are wrong and need corrections or modifications.
3. The method explains some facts which otherwise look mysterious.	3. It could be difficult to transform this explanation into a rigorous proof.
4. It provides a great number of symplectic manifolds and Poisson commuting families of functions.	4. Most completely integrable dynamical systems were discovered earlier by other methods.
5. The method introduces two new fundamental notions: coadjoint orbits and moment maps.	5. The description of coadjoint orbits and their structure is sometimes not an easy problem.



§ 15. The Method of Orbits

At the basis of the method of orbits lies the following "experimental fact": the theory of infinite-dimensional representations of every Lie group is closely connected with a certain special finite-dimensional representation of this group. This representation acts in the dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of the group under study. We will call it a *co-adjoint* or briefly a *K-representation*.[1]

Orbits of a Lie group in the space of a K-representation are symplectic manifolds. They can be interpreted as phase spaces of a Hamiltonian mechanical system for which the given Lie group is the group of symmetries. In 15.2, we shall give a classification of all homogeneous symplectic manifolds with a given group of symmetries.

It turns out that unitary irreducible representations of the group G are connected with orbits of this group in the K-representation. The construction of the representation in an orbit is given in 15.3.

This is a generalization of the procedure of quantization that is used in quantum mechanics. This point of view is explained in more detail in 15.4.

The author sees the significance of the method of orbits not only in the specific theorems obtained by this method, but also in the great collection of simple and intuitive heuristic rules that give the solution of the basic questions of the theory of representations. With the passage of time, these rules will be elevated to the level of strict theorems, but already now their value is indisputable.

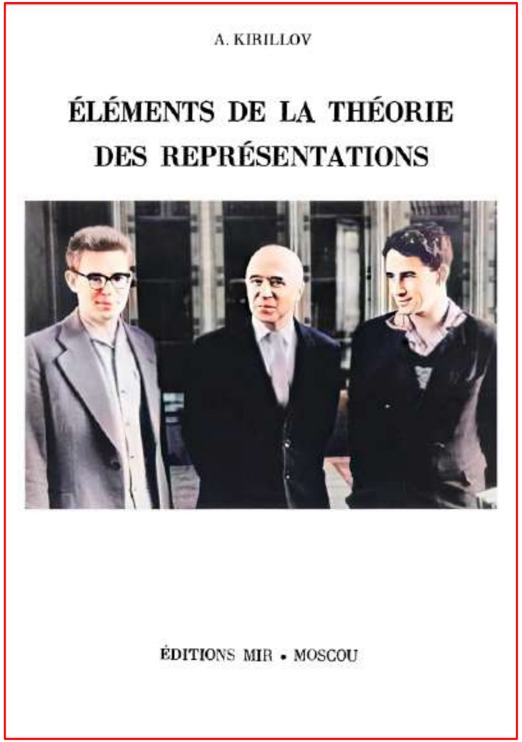
We shall show in 15.5 how the operations of restriction to a subgroup and induction from this subgroup can be described with the aid of the natural projection $p: \mathfrak{g}^* \to \mathfrak{h}^*$, where $\mathfrak{h}$ is the Lie algebra of the subgroup H.

As we shall see in 15.6, generalized characters of irreducible unitary representations admit a simple expression in the form of an integral over the corresponding orbit. In many cases, this allows us to write an explicit expression for the Plancherel measure.

Finally, in 15.7 we show that infinitesimal characters of irreducible unitary representations of a group G can be computed as values of G-invariant polynomials on the corresponding orbits.

15.1. The Co-Adjoint Representation of a Lie Group

Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g}^*$ the dual space to $\mathfrak{g}$. The group G acts in $\mathfrak{g}$ with the aid of the adjoint representation Ad (see 6.3) and in $\mathfrak{g}^*$ with the aid of the co-adjoint representation, or, briefly, the K-representation. If the Lie algebra $\mathfrak{g}$ is realized in the form of the algebra of left-invariant vector-fields on G, then it is natural to realize $\mathfrak{g}^*$ in the form of the space of left-invariant differential forms of the first order on G. The K-representation of the group G acts in the space of 1-forms by right translations.

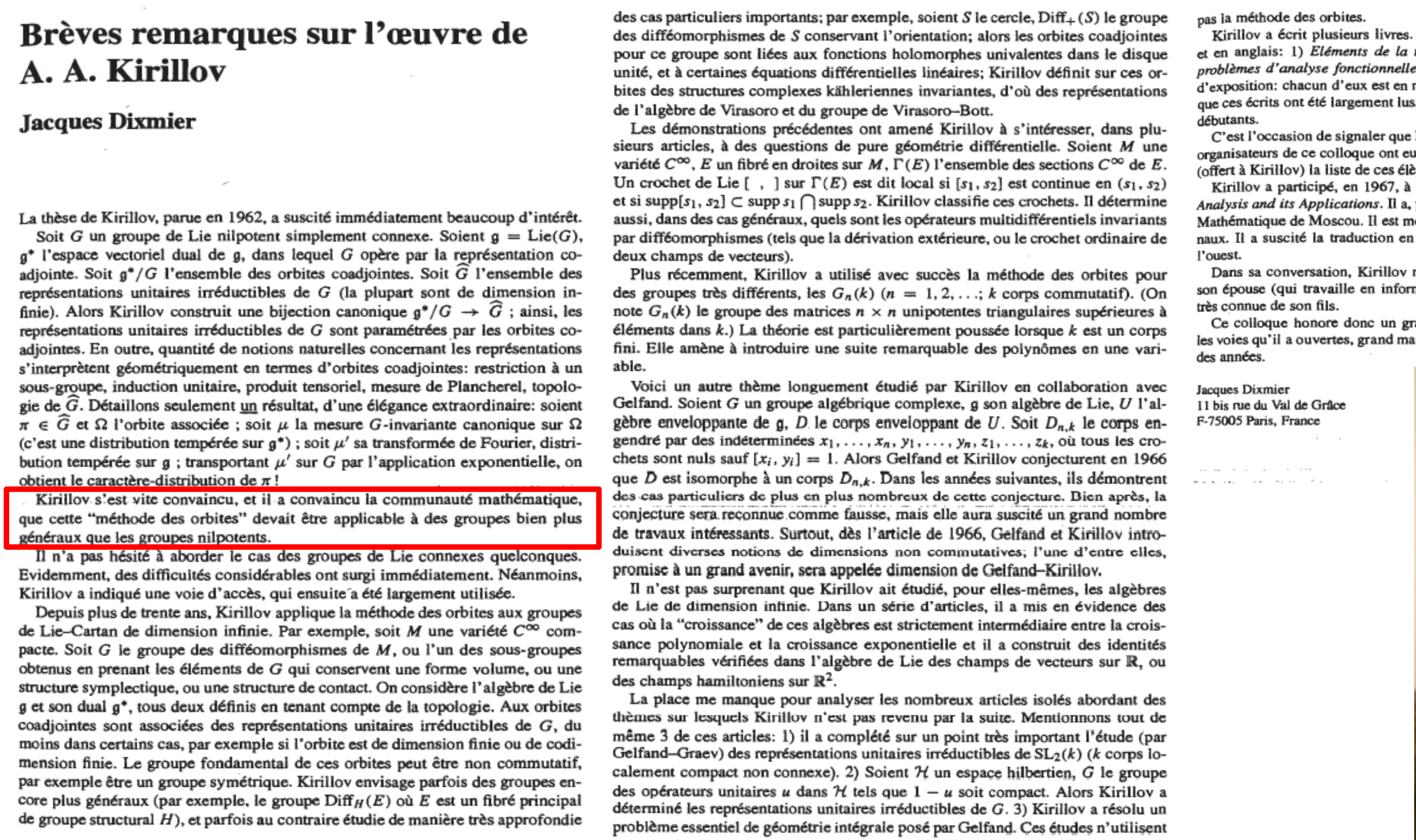


A. KIRILLOV

ÉLÉMENTS DE LA THÉORIE DES REPRÉSENTATIONS

ÉDITIONS MIR • MOSCOU

A.A. Kirillov by Jacques Dixmier



Brèves remarques sur l'œuvre de A. A. Kirillov

Jacques Dixmier

La thèse de Kirillov, parue en 1962, a suscité immédiatement beaucoup d'intérêt.

Soit G un groupe de Lie nilpotent simplement connexe. Soient $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{g}^*$ l'espace vectoriel dual de $\mathfrak{g}$, dans lequel G opère par la représentation coadjointe. Soit $\mathfrak{g}^*/G$ l'ensemble des orbites coadjointes. Soit $\widehat{G}$ l'ensemble des représentations unitaires irréductibles de G (la plupart sont de dimension infinie). Alors Kirillov construit une bijection canonique $\mathfrak{g}^*/G \to \widehat{G}$; ainsi, les représentations unitaires irréductibles de G sont paramétrées par les orbites coadjointes. En outre, quantité de notions naturelles concernant les représentations s'interprètent géométriquement en termes d'orbites coadjointes: restriction à un sous-groupe, induction unitaire, produit tensoriel, mesure de Plancherel, topologie de $\widehat{G}$. Détaillons seulement un résultat, d'une élégance extraordinaire: soient $\pi \in \widehat{G}$ et Ω l'orbite associée ; soit μ la mesure G-invariante canonique sur Ω (c'est une distribution tempérée sur $\mathfrak{g}^*$) ; soit μ' sa transformée de Fourier, distribution tempérée sur $\mathfrak{g}$; transportant μ' sur G par l'application exponentielle, on obtient le caractère-distribution de π !

Kirillov s'est vite convaincu, et il a convaincu la communauté mathématique, que cette "méthode des orbites" devait être applicable à des groupes bien plus généraux que les groupes nilpotents.

Il n'a pas hésité à aborder le cas des groupes de Lie connexes quelconques. Evidemment, des difficultés considérables ont surgi immédiatement. Néanmoins, Kirillov a indiqué une voie d'accès, qui ensuite a été largement utilisée.

Depuis plus de trente ans, Kirillov applique la méthode des orbites aux groupes de Lie–Cartan de dimension infinie. Par exemple, soit M une variété C^∞ compacte. Soit G le groupe des difféomorphismes de M, ou l'un des sous-groupes obtenus en prenant les éléments de G qui conservent une forme volume, ou une structure symplectique, ou une structure de contact. On considère l'algèbre de Lie $\mathfrak{g}$ et son dual $\mathfrak{g}^*$, tous deux définis en tenant compte de la topologie. Aux orbites coadjointes sont associées des représentations unitaires irréductibles de G, du moins dans certains cas, par exemple si l'orbite est de dimension finie ou de codimension finie. Le groupe fondamental de ces orbites peut être non commutatif, par exemple être un groupe symétrique. Kirillov envisage parfois des groupes encore plus généraux (par exemple, le groupe $\mathrm{Diff}_H(E)$ où E est un fibré principal de groupe structural H), et parfois au contraire étudie de manière très approfondie des cas particuliers importants; par exemple, soient S le cercle, $\mathrm{Diff}_+(S)$ le groupe des difféomorphismes de S conservant l'orientation; alors les orbites coadjointes pour ce groupe sont liées aux fonctions holomorphes univalentes dans le disque unité, et à certaines équations différentielles linéaires; Kirillov définit sur ces orbites des structures complexes kähleriennes invariantes, d'où des représentations de l'algèbre de Virasoro et du groupe de Virasoro–Bott.

Les démonstrations précédentes ont amené Kirillov à s'intéresser, dans plusieurs articles, à des questions de pure géométrie différentielle. Soient M une variété C^∞, E un fibré en droites sur M, $\Gamma(E)$ l'ensemble des sections C^∞ de E. Un crochet de Lie $[\ ,\ ]$ sur $\Gamma(E)$ est dit local si $[s_1, s_2]$ est continue en (s_1, s_2) et si $\mathrm{supp}[s_1, s_2] \subset \mathrm{supp}\, s_1 \bigcap \mathrm{supp}\, s_2$. Kirillov classifie ces crochets. Il détermine aussi, dans des cas généraux, quels sont les opérateurs multidifférentiels invariants par difféomorphismes (tels que la dérivation extérieure, ou le crochet ordinaire de deux champs de vecteurs).

Plus récemment, Kirillov a utilisé avec succès la méthode des orbites pour des groupes très différents, les $G_n(k)$ ($n = 1, 2, \ldots$; k corps commutatif). (On note $G_n(k)$ le groupe des matrices $n \times n$ unipotentes triangulaires supérieures à éléments dans k.) La théorie est particulièrement poussée lorsque k est un corps fini. Elle amène à introduire une suite remarquable des polynômes en une variable.

Voici un autre thème longuement étudié par Kirillov en collaboration avec Gelfand. Soient G un groupe algébrique complexe, $\mathfrak{g}$ son algèbre de Lie, U l'algèbre enveloppante de $\mathfrak{g}$, D le corps enveloppant de U. Soit $D_{n,k}$ le corps engendré par des indéterminées $x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_k$, où tous les crochets sont nuls sauf $[x_i, y_i] = 1$. Alors Gelfand et Kirillov conjecturent en 1966 que D est isomorphe à un corps $D_{n,k}$. Dans les années suivantes, ils démontrent des cas particuliers de plus en plus nombreux de cette conjecture. Bien après, la conjecture sera reconnue comme fausse, mais elle aura suscité un grand nombre de travaux intéressants. Surtout, dès l'article de 1966, Gelfand et Kirillov introduisent diverses notions de dimensions non commutatives; l'une d'entre elles, promise à un grand avenir, sera appelée dimension de Gelfand–Kirillov.

Il n'est pas surprenant que Kirillov ait étudié, pour elles-mêmes, les algèbres de Lie de dimension infinie. Dans un série d'articles, il a mis en évidence des cas où la "croissance" de ces algèbres est strictement intermédiaire entre la croissance polynomiale et la croissance exponentielle et il a construit des identités remarquables vérifiées dans l'algèbre de Lie des champs de vecteurs sur $\mathbb{R}$, ou des champs hamiltoniens sur $\mathbb{R}^2$.

La place me manque pour analyser les nombreux articles isolés abordant des thèmes sur lesquels Kirillov n'est pas revenu par la suite. Mentionnons tout de même 3 de ces articles: 1) il a complété sur un point très important l'étude (par Gelfand–Graev) des représentations unitaires irréductibles de $\mathrm{SL}_2(k)$ (k corps localement compact non connexe). 2) Soient $\mathcal{H}$ un espace hilbertien, G le groupe des opérateurs unitaires u dans $\mathcal{H}$ tels que $1 - u$ soit compact. Alors Kirillov a déterminé les représentations unitaires irréductibles de G. 3) Kirillov a résolu un problème essentiel de géométrie intégrale posé par Gelfand. Ces études n'utilisent pas la méthode des orbites.

Kirillov a écrit plusieurs livres. Deux d'entre eux ont été traduits en français et en anglais: 1) *Eléments de la théorie des représentations*, 2) *Théorèmes et problèmes d'analyse fonctionnelle*. Il a d'autre part rédigé beaucoup d'articles d'exposition: chacun d'eux est en réalité un court livre. La clarté des exposés fait que ces écrits ont été largement lus, et notamment, avec profit, par des chercheurs débutants.

C'est l'occasion de signaler que Kirillov a eu de nombreux élèves de thèse. Les organisateurs de ce colloque ont eu la bonne idée de faire imprimer sur un T-shirt (offert à Kirillov) la liste de ces élèves: 57 noms, un record !

Kirillov a participé, en 1967, à la fondation d'un journal célèbre: *Functional Analysis and its Applications*. Il a, pendant 4 ans, été vice-président de la Société Mathématique de Moscou. Il est membre de comités éditoriaux de plusieurs journaux. Il a suscité la traduction en russe de nombreux livres et articles édités à l'ouest.

Dans sa conversation, Kirillov mentionne le soutien constant qu'il a reçu de son épouse (qui travaille en informatique). Il est fier de l'œuvre mathématique très connue de son fils.

Ce colloque honore donc un grand mathématicien: grand mathématicien par les voies qu'il a ouvertes, grand mathématicien par l'influence qu'il exerce depuis des années.

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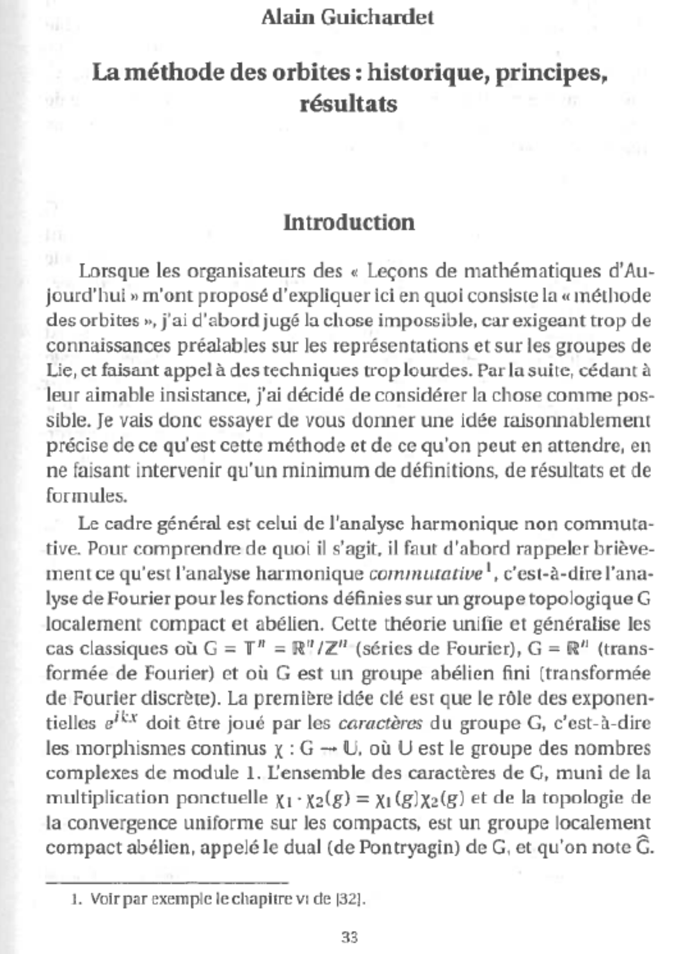
Jean-Marie SOURIAU

La méthodes des orbites: outil essentiel de l'analyse harmonique non-commutative (Alain Guichardet, leçons de mathématiques d'aujourd'hui, Vol.4)



Alain Guichardet

La méthode des orbites : historique, principes, résultats

Introduction

Lorsque les organisateurs des « Leçons de mathématiques d'Aujourd'hui » m'ont proposé d'expliquer ici en quoi consiste la « méthode des orbites », j'ai d'abord jugé la chose impossible, car exigeant trop de connaissances préalables sur les représentations et sur les groupes de Lie, et faisant appel à des techniques trop lourdes. Par la suite, cédant à leur aimable insistance, j'ai décidé de considérer la chose comme possible. Je vais donc essayer de vous donner une idée raisonnablement précise de ce qu'est cette méthode et de ce qu'on peut en attendre, en ne faisant intervenir qu'un minimum de définitions, de résultats et de formules.

Le cadre général est celui de l'analyse harmonique non commutative. Pour comprendre de quoi il s'agit, il faut d'abord rappeler brièvement ce qu'est l'analyse harmonique *commutative*[1], c'est-à-dire l'analyse de Fourier pour les fonctions définies sur un groupe topologique G localement compact et abélien. Cette théorie unifie et généralise les cas classiques où $G = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ (séries de Fourier), $G = \mathbb{R}^n$ (transformée de Fourier) et où G est un groupe abélien fini (transformée de Fourier discrète). La première idée clé est que le rôle des exponentielles e^{ikx} doit être joué par les *caractères* du groupe G, c'est-à-dire les morphismes continus $\chi : G \to \mathbb{U}$, où $\mathbb{U}$ est le groupe des nombres complexes de module 1. L'ensemble des caractères de G, muni de la multiplication ponctuelle $\chi_1 \cdot \chi_2(g) = \chi_1(g)\chi_2(g)$ et de la topologie de la convergence uniforme sur les compacts, est un groupe localement compact abélien, appelé le dual (de Pontryagin) de G, et qu'on note $\widehat{G}$.

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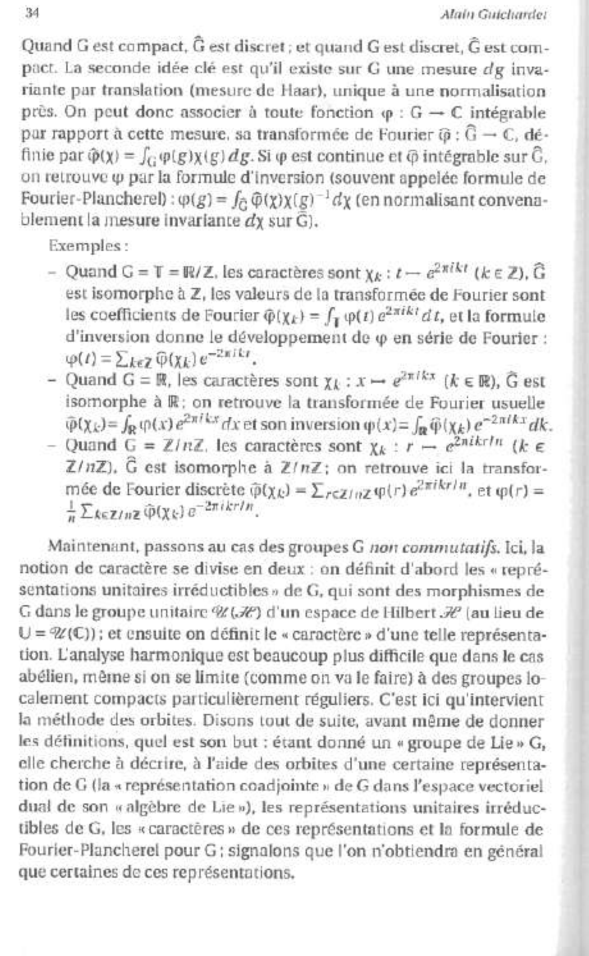
[1] Voir par exemple le chapitre VI de [32].

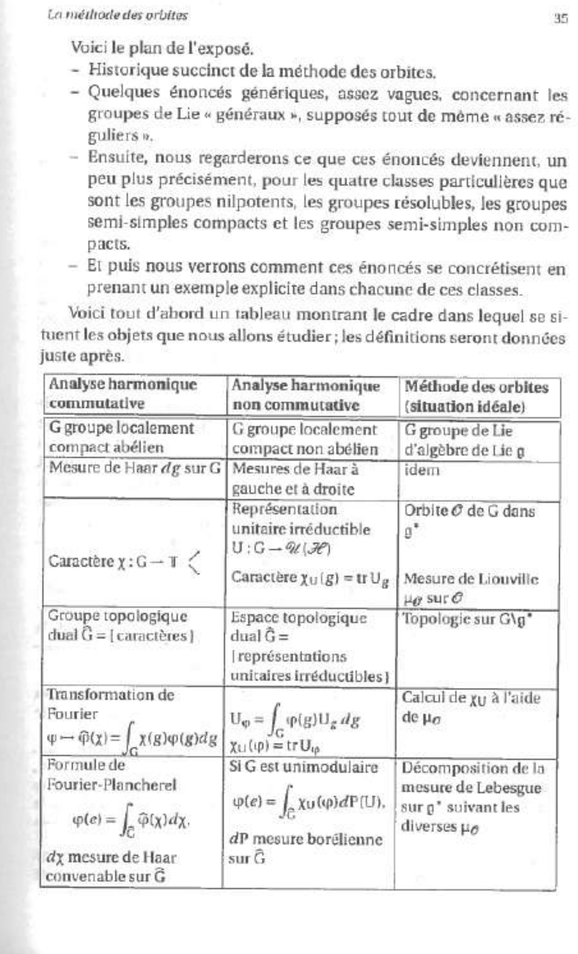


Quand G est compact, $\widehat{G}$ est discret ; et quand G est discret, $\widehat{G}$ est compact. La seconde idée clé est qu'il existe sur G une mesure dg invariante par translation (mesure de Haar), unique à une normalisation près. On peut donc associer à toute fonction $\varphi : G \to \mathbb{C}$ intégrable par rapport à cette mesure, sa transformée de Fourier $\widehat{\varphi} : \widehat{G} \to \mathbb{C}$, définie par $\widehat{\varphi}(\chi) = \int_G \varphi(g)\chi(g)\,dg$. Si φ est continue et $\widehat{\varphi}$ intégrable sur $\widehat{G}$, on retrouve φ par la formule d'inversion (souvent appelée formule de Fourier-Plancherel) : $\varphi(g) = \int_{\widehat{G}} \widehat{\varphi}(\chi)\chi(g)^{-1}\,d\chi$ (en normalisant convenablement la mesure invariante $d\chi$ sur $\widehat{G}$).

Exemples :

- Quand $G = \mathbb{T} = \mathbb{R}/\mathbb{Z}$, les caractères sont $\chi_k : t \mapsto e^{2\pi ikt}$ ($k \in \mathbb{Z}$), $\widehat{G}$ est isomorphe à $\mathbb{Z}$, les valeurs de la transformée de Fourier sont les coefficients de Fourier $\widehat{\varphi}(\chi_k) = \int_{\mathbb{T}} \varphi(t)\,e^{2\pi ikt}\,dt$, et la formule d'inversion donne le développement de φ en série de Fourier : $\varphi(t) = \sum_{k\in\mathbb{Z}} \widehat{\varphi}(\chi_k)\,e^{-2\pi ikt}$.
- Quand $G = \mathbb{R}$, les caractères sont $\chi_k : x \mapsto e^{2\pi ikx}$ ($k \in \mathbb{R}$), $\widehat{G}$ est isomorphe à $\mathbb{R}$; on retrouve la transformée de Fourier usuelle $\widehat{\varphi}(\chi_k) = \int_{\mathbb{R}} \varphi(x)\,e^{2\pi ikx}\,dx$ et son inversion $\varphi(x) = \int_{\mathbb{R}} \widehat{\varphi}(\chi_k)\,e^{-2\pi ikx}\,dk$.
- Quand $G = \mathbb{Z}/n\mathbb{Z}$, les caractères sont $\chi_k : r \mapsto e^{2\pi ikr/n}$ ($k \in \mathbb{Z}/n\mathbb{Z}$), $\widehat{G}$ est isomorphe à $\mathbb{Z}/n\mathbb{Z}$; on retrouve ici la transformée de Fourier discrète $\widehat{\varphi}(\chi_k) = \sum_{r\in\mathbb{Z}/n\mathbb{Z}} \varphi(r)\,e^{2\pi ikr/n}$, et $\varphi(r) = \frac{1}{n}\sum_{k\in\mathbb{Z}/n\mathbb{Z}} \widehat{\varphi}(\chi_k)\,e^{-2\pi ikr/n}$.

Maintenant, passons au cas des groupes G *non commutatifs*. Ici, la notion de caractère se divise en deux : on définit d'abord les « représentations unitaires irréductibles » de G, qui sont des morphismes de G dans le groupe unitaire $\mathcal{U}(\mathcal{H})$ d'un espace de Hilbert $\mathcal{H}$ (au lieu de $\mathbb{U} = \mathcal{U}(\mathbb{C})$) ; et ensuite on définit le « caractère » d'une telle représentation. L'analyse harmonique est beaucoup plus difficile que dans le cas abélien, même si on se limite (comme on va le faire) à des groupes localement compacts particulièrement réguliers. C'est ici qu'intervient la méthode des orbites. Disons tout de suite, avant même de donner les définitions, quel est son but : étant donné un « groupe de Lie » G, elle cherche à décrire, à l'aide des orbites d'une certaine représentation de G (la « représentation coadjointe » de G dans l'espace vectoriel dual de son « algèbre de Lie »), les représentations unitaires irréductibles de G, les « caractères » de ces représentations et la formule de Fourier-Plancherel pour G ; signalons que l'on n'obtiendra en général que certaines de ces représentations.



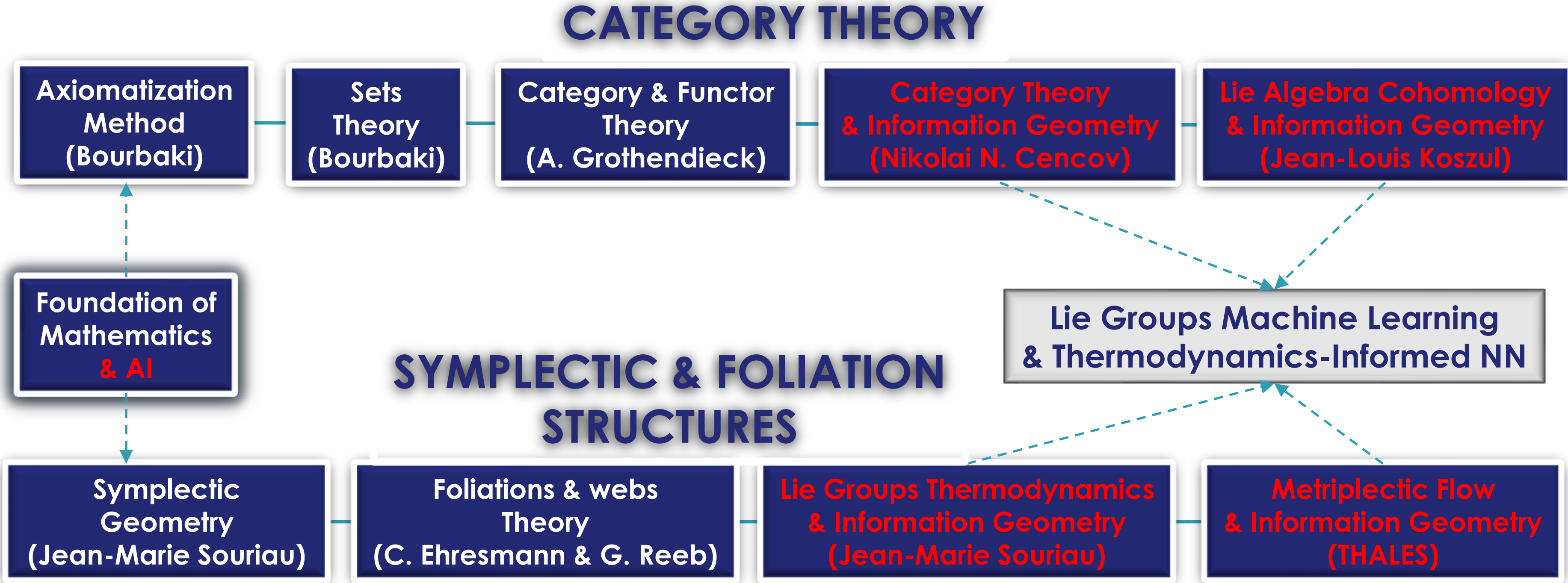
Voici le plan de l'exposé.

- Historique succinct de la méthode des orbites.
- Quelques énoncés génériques, assez vagues, concernant les groupes de Lie « généraux », supposés tout de même « assez réguliers ».
- Ensuite, nous regarderons ce que ces énoncés deviennent, un peu plus précisément, pour les quatre classes particulières que sont les groupes nilpotents, les groupes résolubles, les groupes semi-simples compacts et les groupes semi-simples non compacts.
- Et puis nous verrons comment ces énoncés se concrétisent en prenant un exemple explicite dans chacune de ces classes.

Voici tout d'abord un tableau montrant le cadre dans lequel se situent les objets que nous allons étudier ; les définitions seront données juste après.

Analyse harmonique commutative	**Analyse harmonique non commutative**	**Méthode des orbites (situation idéale)**
G groupe localement compact abélien	G groupe localement compact non abélien	G groupe de Lie d'algèbre de Lie $\mathfrak{g}$
Mesure de Haar dg sur G	Mesures de Haar à gauche et à droite	idem
Caractère $\chi : G \to \mathbb{T}$ <	Représentation unitaire irréductible $U : G \to \mathcal{U}(\mathcal{H})$ Caractère $\chi_U(g) = \mathrm{tr}\, U_g$	Orbite $\mathcal{O}$ de G dans $\mathfrak{g}^*$ Mesure de Liouville $\mu_{\mathcal{O}}$ sur $\mathcal{O}$
Groupe topologique dual $\widehat{G}$ = {caractères}	Espace topologique dual $\widehat{G}$ = {représentations unitaires irréductibles}	Topologie sur $G\backslash\mathfrak{g}^*$
Transformation de Fourier $\varphi \mapsto \widehat{\varphi}(\chi) = \int_G \chi(g)\varphi(g)dg$	$U_\varphi = \int_G \varphi(g)U_g\,dg$ $\chi_U(\varphi) = \mathrm{tr}\, U_\varphi$	Calcul de χ_U à l'aide de $\mu_{\mathcal{O}}$
Formule de Fourier-Plancherel $\varphi(e) = \int_{\widehat{G}} \widehat{\varphi}(\chi)d\chi$, $d\chi$ mesure de Haar convenable sur $\widehat{G}$	Si G est unimodulaire $\varphi(e) = \int_{\widehat{G}} \chi_U(\varphi)dP(U)$, dP mesure borélienne sur $\widehat{G}$	Décomposition de la mesure de Lebesgue sur $\mathfrak{g}^*$ suivant les diverses $\mu_{\mathcal{O}}$

Mathematical Foundations of AI



CATEGORY THEORY

Axiomatization Method (Bourbaki)

Sets Theory (Bourbaki)

Category & Functor Theory (A. Grothendieck)

Category Theory & Information Geometry (Nikolai N. Cencov)

Lie Algebra Cohomology & Information Geometry (Jean-Louis Koszul)

Foundation of Mathematics & AI

Lie Groups Machine Learning & Thermodynamics-Informed NN

SYMPLECTIC & FOLIATION STRUCTURES

Symplectic Geometry (Jean-Marie Souriau)

Foliations & webs Theory (C. Ehresmann & G. Reeb)

Lie Groups Thermodynamics & Information Geometry (Jean-Marie Souriau)

Metriplectic Flow & Information Geometry (THALES)

AI Mathematical Foundations: Symplectic Foliations

Lie Groups & Lie Algebra Representation Theory

Lie Algebra Cohomology, Co-adjoint Orbits, Cocycle, Quivers

Symplectic Geometry & Poisson Geometry

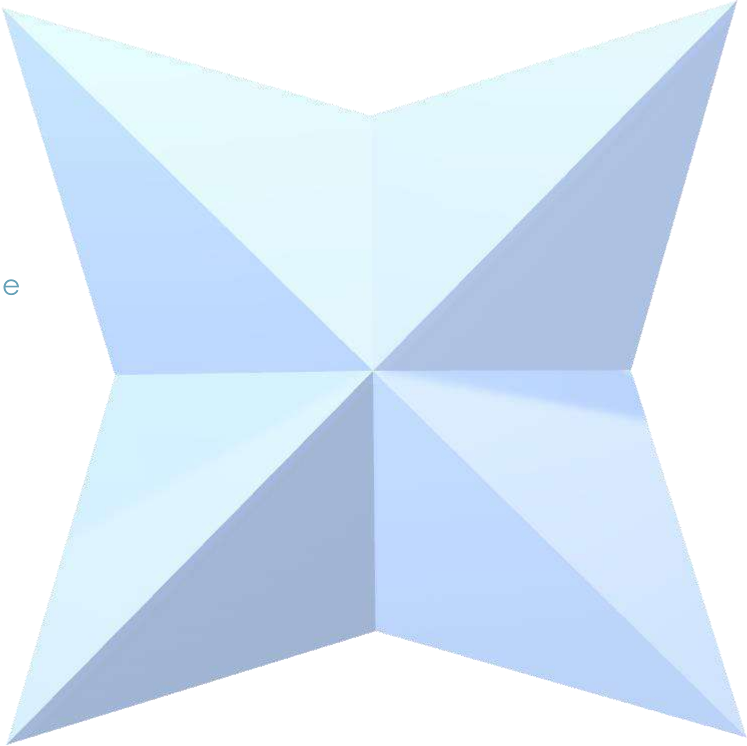
Symplectic Manifold associated to Lie Group Co-adjoint Orbit by KKS 2 Form, Moment Map, Souriau's Maxwell Principle

Integrable System

complete integrability, Liouville integrability, Action-Angles Coordinates, Symplectic Integrators

Foliation Theory

Coisotropic Polar, Bifibration, Bifoliation, Transverse Poisson Structure, Symplectic Foliation, Riemannian Foliation, webs



Information Geometry

Statistical Manifold, Koszul-Fisher Metric, Natural Gradient, Natural Langevin Dynamics, Coordinate Free Neural Network, Hessian Structures, Flat connexion without torsion

Metriplectic Flow

GENERIC Flow, Poisson Bracket, Metric Flow Bracket, 1st & 2nd Thermodynamics Principles, Onsager-Casimir Relations, Dissipation, Transverse Foliations of Entropy and Energy levels sets

Calculus of Variations

Geometrization of Noether Theorem, Poincaré-Cartan Integral Invariant, Euler-Poincaré Equation

Souriau's Lie Groups Thermodynamics

Covariant Gibbs Density, Planck Temperature in Lie Algebra, Heat in dual Lie Algebra, Entropy as Casimir function, Souriau-Fisher Metric

Metriplectic Flow on Symplectic Foliation & Transverse Metric Foliation

**Foliation Leaves =
Level Sets of Energy**

**Foliation Leaves =
Level Sets of Entropy**

METRIC FOLIATION $\mathfrak{I}^{\perp}\perp\mathfrak{I}$ **SYMPLECTIC FOLIATION**

**1st Principle of Thermodynamics
Preservation of Energy**

**2nd Principle of Thermodynamics
Entropy Production**

METRIPLECTIC FLOW

$$\frac{dF}{dt}=\{F,H\}+(F,S)$$

**Non-dissipative
Entropy = Constant**

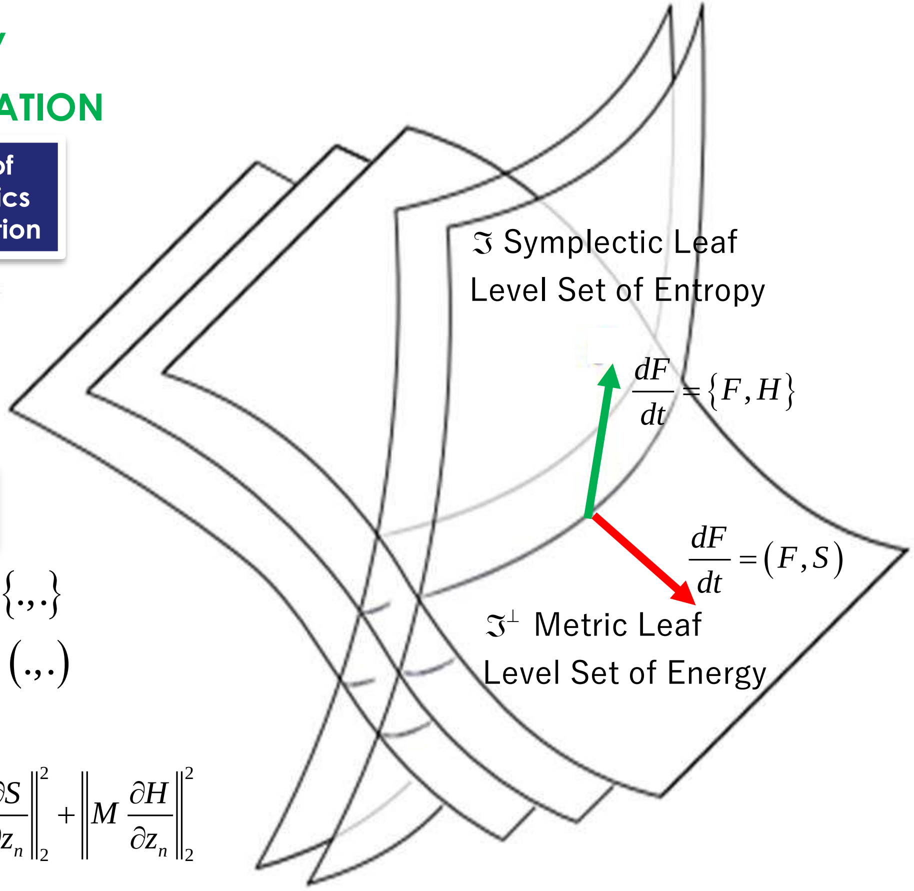
**Dissipative
Energy = Constant**

$\{F,S\}=0\ \ \forall F : S$ Entropy Casimir Function for $\{.,.\}$

$(F,H)=0\ \ \forall F : H$ Energy Casimir Function for $(.,.)$

Thermodynamics-Informed Neural Networks

$$\Lambda=\frac{1}{N_{batch}}\sum_{n=0}^{N_{batch}}\left(\lambda\Lambda_n^{data}+\Lambda_n^{\deg}\right)\qquad \Lambda_n^{data}=\left\|\frac{dz^{GT}}{dt}-\frac{dz^{net}}{dt}\right\|_2^2\qquad \Lambda_n^{\deg}=\left\|L\frac{\partial S}{\partial z_n}\right\|_2^2+\left\|M\frac{\partial H}{\partial z_n}\right\|_2^2$$



GEOMETRIC DEFINITION OF ENTROPY

ENTROPY DEFINITION: Entropy is Invariant Casimir Function on Leaves of Symplectic Foliation associated to Coadjoint Orbits generated via Moment Map of Symmetry Group acting on the System.

$$\boxed{\{S,H\}_{\tilde{\Theta}}(Q)=0}$$

$$\boxed{ad^{*}_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0}$$

$$\beta\in Ker\tilde{\Theta}_{\beta}\Rightarrow\left\langle Q,\left[\frac{\partial S}{\partial Q},Z\right]\right\rangle+\tilde{\Theta}\left(\frac{\partial S}{\partial Q},Z\right)=0$$

Sadi Carnot Cycle on Souriau's Symplectic Leaves

$$\left\langle Q,[\beta,Z]\right\rangle+\tilde{\Theta}(\beta,Z)=0$$

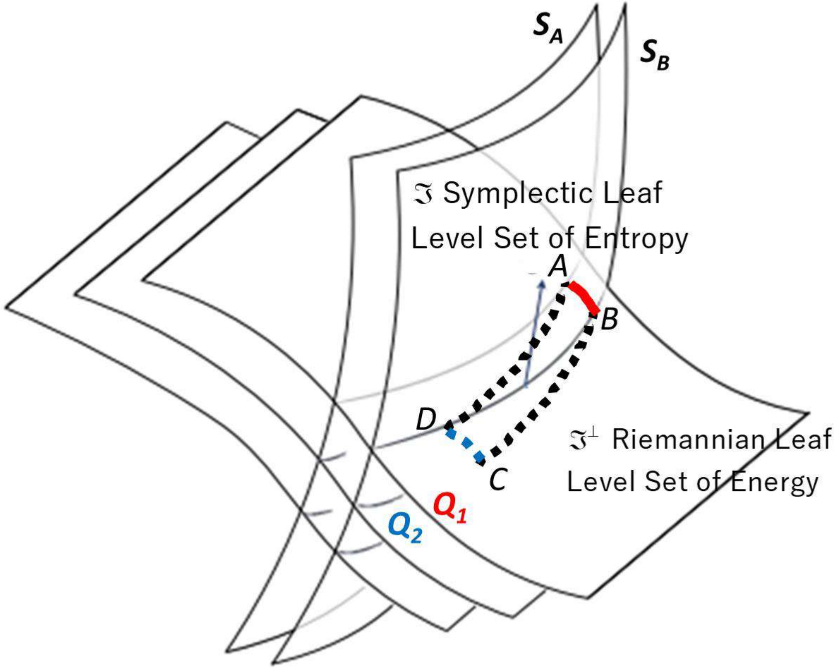
$$\{S,H\}_{\tilde{\Theta}}(Q)=0$$

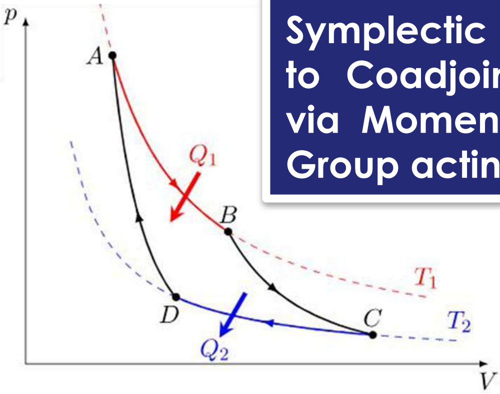
$$\beta\in Ker\tilde{\Theta}_{\beta}\Rightarrow\left\langle Q,\left[\frac{\partial S}{\partial Q},Z\right]\right\rangle+\tilde{\Theta}\left(\frac{\partial S}{\partial Q},Z\right)=0$$

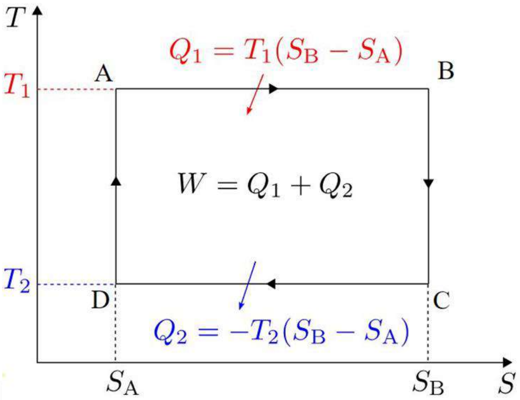
$$ad^{*}_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0$$

ENTROPY DEFINITION:

Entropy is Invariant Casimir Function on Leaves of Symplectic Foliation associated to Coadjoint Orbits generated via Moment Map of Symmetry Group acting on the System.







THALES
Building a future we can all trust

METRIPLECTIC MODEL

The symmetric dissipative metriplectic bracket as Euclidean metric tensor on the symplectic leaves foliated by the Casimir invariants

- ***Sato observed that the canonical form of the symmetric dissipative part of the metriplectic bracket is identified in terms of a 'canonical metric tensor' corresponding to an Euclidean metric tensor on the symplectic leaves foliated by the Casimir invariants.***
- It makes the link with Symplectic model of Lie groups Thermodynamics. A single generating Massieu function $\Phi = \langle \beta, Q \rangle - S$ **is sufficient to generate the dynamics by the action of the metriplectic bracket:**

$$\frac{dF}{dt} = \{\{F,\Phi\}\} = \{F,Q\} + (F,S) = \beta^{-1}\{F,\Phi\} - (F,\Phi)$$

$$\{F,S\} = 0 \quad \text{and} \quad (F,Q) = 0$$

$$H = W + Q$$

A Single Generating Function Generates the Metriplectic Dynamics:

François Massieu Characteristic Function

Thermodynamics Triptych : symplectic & transverse metric foliatons

DUAL FISHER METRIC
HESSIAN OF ENTROPY

DISSIPATIVE

Transverse Riemanian Foliation

Baptiste COQUINOT

Metriplectic Flow & Onsager-Casimir Relations

Koszul Information Geometry

Reeb's Foliation Theory: Symplectic Foliation, Transverse Metric Foliation

Charles EHRESMANN

Georges Henri REEB

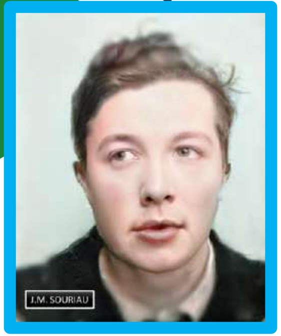
Symplectic Foliation

Souriau's Lie Groups Thermodynamics with Entropy as Casimir Function

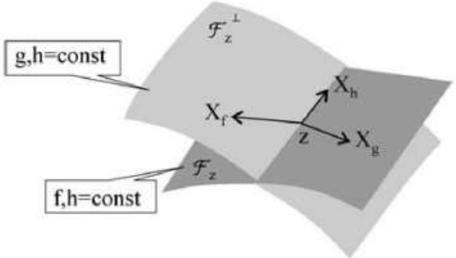
NON-DISSIPATIVE

FISHER METRIC
HESSIAN OF MASSIEU
KKS 2-FORM

Jean-Louis KOSZUL

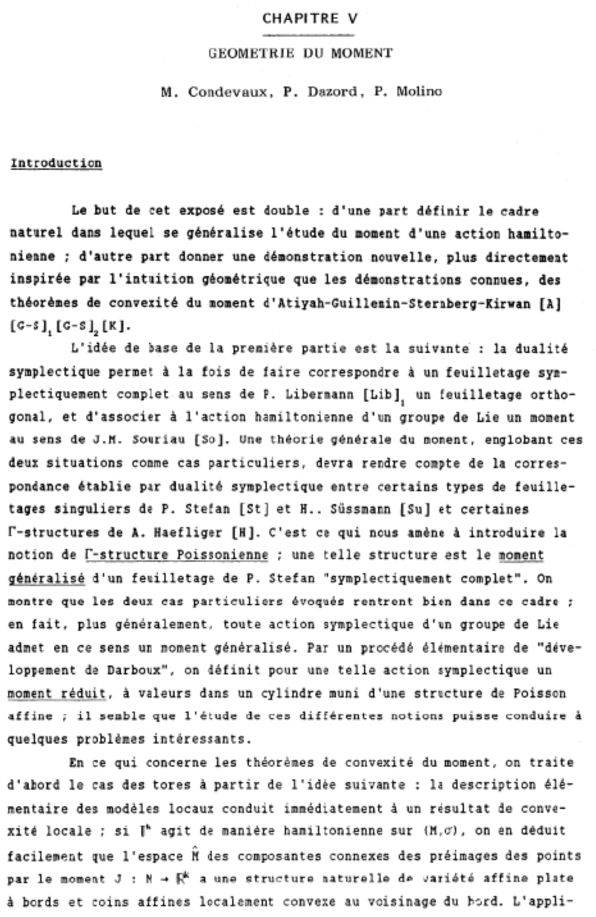


Jean-Marie SOURIAU



Moment Map Geometry & Libermann Foliation: Condevaux, Dazord, Molino

Séminaire Sud-Rhodanien



CHAPITRE V

GEOMETRIE DU MOMENT

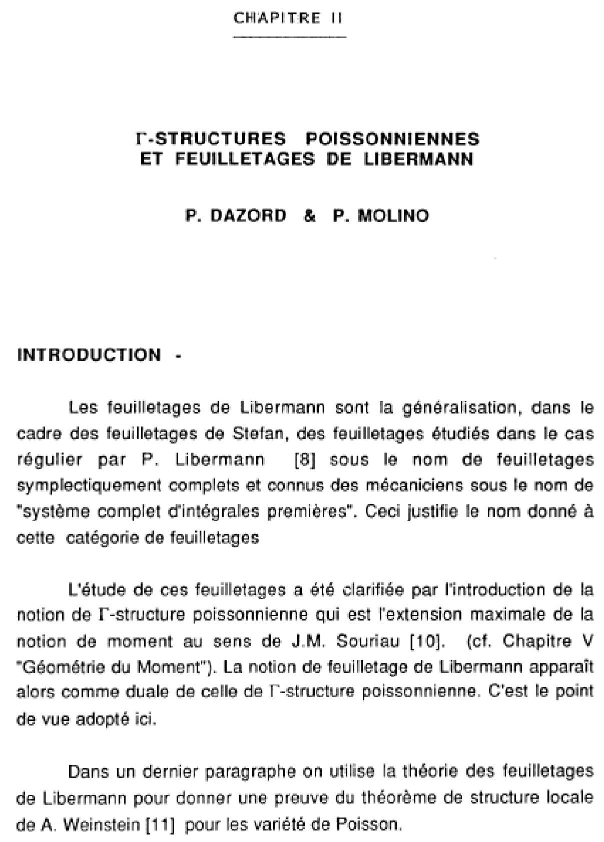
M. Condevaux, P. Dazord, P. Molino

Introduction

Le but de cet exposé est double : d'une part définir le cadre naturel dans lequel se généralise l'étude du moment d'une action hamiltonienne ; d'autre part donner une démonstration nouvelle, plus directement inspirée par l'intuition géométrique que les démonstrations connues, des théorèmes de convexité du moment d'Atiyah-Guillemin-Sternberg-Kirwan [A] $[G\text{-}S]_1$ $[G\text{-}S]_2$ [K].

L'idée de base de la première partie est la suivante : la dualité symplectique permet à la fois de faire correspondre à un feuilletage symplectiquement complet au sens de P. Libermann $[Lib]_1$ un feuilletage orthogonal, et d'associer à l'action hamiltonienne d'un groupe de Lie un moment au sens de J.M. Souriau [So]. Une théorie générale du moment, englobant ces deux situations comme cas particuliers, devra rendre compte de la correspondance établie par dualité symplectique entre certains types de feuilletages singuliers de P. Stefan [St] et H.. Süssmann [Su] et certaines Γ-structures de A. Haefliger [H]. C'est ce qui nous amène à introduire la notion de *Γ-structure Poissonienne* ; une telle structure est le *moment généralisé* d'un feuilletage de P. Stefan "symplectiquement complet". On montre que les deux cas particuliers évoqués rentrent bien dans ce cadre ; en fait, plus généralement, toute action symplectique d'un groupe de Lie admet en ce sens un moment généralisé. Par un procédé élémentaire de "développement de Darboux", on définit pour une telle action symplectique un *moment réduit*, à valeurs dans un cylindre muni d'une structure de Poisson affine ; il semble que l'étude de ces différentes notions puisse conduire à quelques problèmes intéressants.

En ce qui concerne les théorèmes de convexité du moment, on traite d'abord le cas des tores à partir de l'idée suivante : la description élémentaire des modèles locaux conduit immédiatement à un résultat de convexité locale ; si T^k agit de manière hamiltonienne sur (M,σ), on en déduit facilement que l'espace $\hat{M}$ des composantes connexes des préimages des points par le moment $J : M \to \mathbb{R}^k$ a une structure naturelle de variété affine plate à bords et coins affines localement convexe au voisinage du bord. L'appli-



CHAPITRE II

Γ-STRUCTURES POISSONNIENNES ET FEUILLETAGES DE LIBERMANN

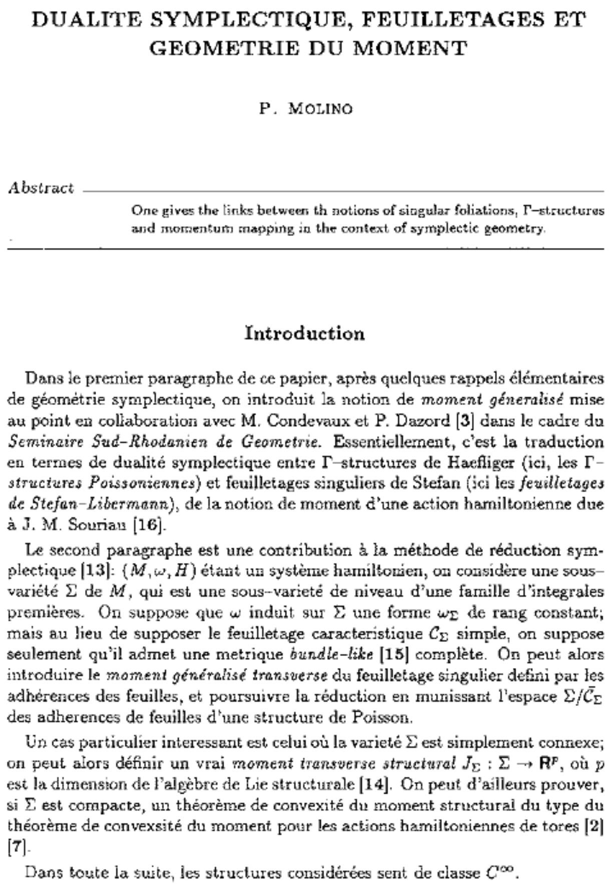
P. DAZORD & P. MOLINO

INTRODUCTION -

Les feuilletages de Libermann sont la généralisation, dans le cadre des feuilletages de Stefan, des feuilletages étudiés dans le cas régulier par P. Libermann [8] sous le nom de feuilletages symplectiquement complets et connus des mécaniciens sous le nom de "système complet d'intégrales premières". Ceci justifie le nom donné à cette catégorie de feuilletages

L'étude de ces feuilletages a été clarifiée par l'introduction de la notion de Γ-structure poissonnienne qui est l'extension maximale de la notion de moment au sens de J.M. Souriau [10]. (cf. Chapitre V "Géométrie du Moment"). La notion de feuilletage de Libermann apparaît alors comme duale de celle de Γ-structure poissonnienne. C'est le point de vue adopté ici.

Dans un dernier paragraphe on utilise la théorie des feuilletages de Libermann pour donner une preuve du théorème de structure locale de A. Weinstein [11] pour les variété de Poisson.



DUALITE SYMPLECTIQUE, FEUILLETAGES ET GEOMETRIE DU MOMENT

P. MOLINO

Abstract

One gives the links between th notions of singular foliations, Γ–structures and momentum mapping in the context of symplectic geometry.

Introduction

Dans le premier paragraphe de ce papier, après quelques rappels élémentaires de géométrie symplectique, on introduit la notion de *moment géneralisé* mise au point en collaboration avec M. Condevaux et P. Dazord [3] dans le cadre du *Seminaire Sud–Rhodanien de Geometrie.* Essentiellement, c'est la traduction en termes de dualité symplectique entre Γ–structures de Haefliger (ici, les *Γ–structures Poissoniennes*) et feuilletages singuliers de Stefan (ici les *feuilletages de Stefan–Libermann*), de la notion de moment d'une action hamiltonienne due à J. M. Souriau [16].

Le second paragraphe est une contribution à la méthode de réduction symplectique [13]: (M,ω,H) étant un système hamiltonien, on considère une sous–variété Σ de M, qui est une sous–varieté de niveau d'une famille d'integrales premières. On suppose que ω induit sur Σ une forme ω_Σ de rang constant; mais au lieu de supposer le feuilletage caracteristique C_Σ simple, on suppose seulement qu'il admet une metrique *bundle–like* [15] complète. On peut alors introduire le *moment généralisé transverse* du feuilletage singulier defini par les adhérences des feuilles, et poursuivre la réduction en munissant l'espace $\Sigma/\bar{C}_\Sigma$ des adherences de feuilles d'une structure de Poisson.

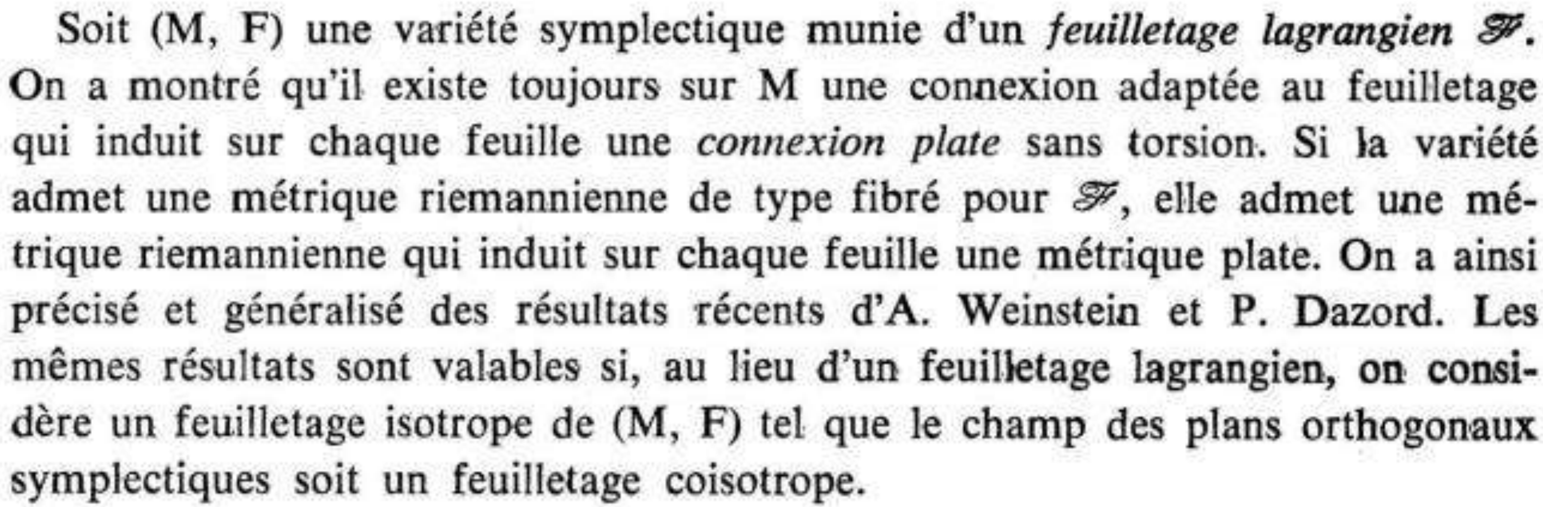
Un cas particulier interessant est celui où la varieté Σ est simplement connexe; on peut alors définir un vrai *moment transverse structural* $J_\Sigma : \Sigma \to \mathbf{R}^p$, où p est la dimension de l'algèbre de Lie structurale [14]. On peut d'ailleurs prouver, si Σ est compacte, un théorème de convexité du moment structural du type du théorème de convexsité du moment pour les actions hamiltoniennes de tores [2] [7].

Dans toute la suite, les structures considérées sont de classe C^∞.

André Lichnerowicz, Collège de France

Cours 1982-1983 – Étude systématique des rapports entre feuilletages et variétés de Poisson | Étude de cohomologies d'algèbres de Lie attachées à une variété de contact

- https://www.college-de-france.fr/sites/default/files/media/document/2023-05/1982-1983_lichnerowicz.pdf



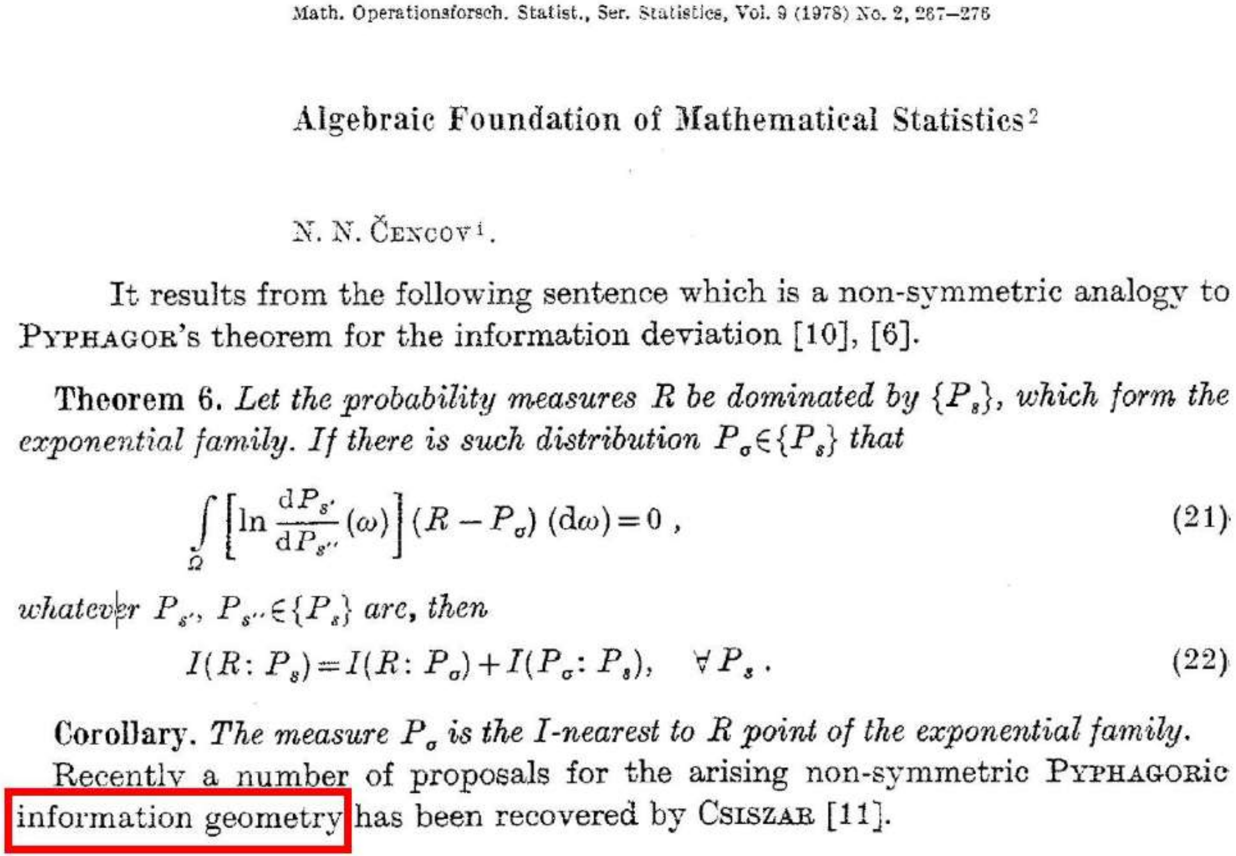
Soit (M, F) une variété symplectique munie d'un *feuilletage lagrangien* $\mathscr{F}$. On a montré qu'il existe toujours sur M une connexion adaptée au feuilletage qui induit sur chaque feuille une *connexion plate* sans torsion. Si la variété admet une métrique riemannienne de type fibré pour $\mathscr{F}$, elle admet une métrique riemannienne qui induit sur chaque feuille une métrique plate. On a ainsi précisé et généralisé des résultats récents d'A. Weinstein et P. Dazord. Les mêmes résultats sont valables si, au lieu d'un feuilletage lagrangien, on considère un feuilletage isotrope de (M, F) tel que le champ des plans orthogonaux symplectiques soit un feuilletage coisotrope.

Transversally oriented foliation & the Godbillon–Vey characteristic class

GODBILLON–VEY CLASSES OF SYMPLECTIC FOLIATIONS by Kentaro Mikami

- **Each transversally oriented foliation has the Godbillon–Vey characteristic class, and regular Poisson structures define symplectic foliations**.
- K. Mikami has given a new interpretation and the explicit formula for a representative of the Godbillon–Vey characteristic classes of symplectic foliations in the context of Poisson geometry.
- For each transversely oriented foliation, we have the famous Godbillon-Vey characteristic class. **When the symplectic foliations of regular Poisson structures are transversely oriented, they have the Godbillon-Vey characteristic classes**. K. Mikami has given a formula defining their **Godbillon-Vey classes in terms of Poisson structure**.
- **Transversely oriented foliations have secondary characteristic classes which are called the Godbillon-Vey class.**
  - Reference: **J.-L. Koszul, Crochet de Schouten-Nijenhuis et cohomologie, in ‘Elie Cartan et les Mathématiques d’aujourd’hui’, 257-271; Société Math. de France, Astérisque, hors série, 1985.**

Axiomatization of Information Geometry via Category Theory
Nikolai Nikolaevich Chentsov in 60's



Math. Operationsforsch. Statist., Ser. Statistics, Vol. 9 (1978) No. 2, 267–276

Algebraic Foundation of Mathematical Statistics[2]

N. N. Čencov[1].

It results from the following sentence which is a non-symmetric analogy to Pyphagor's theorem for the information deviation [10], [6].

Theorem 6. *Let the probability measures R be dominated by $\{P_s\}$, which form the exponential family. If there is such distribution $P_\sigma \in \{P_s\}$ that*

$$\int_\Omega \left[\ln \frac{\mathrm{d}P_{s'}}{\mathrm{d}P_{s''}}(\omega)\right](R - P_\sigma)(\mathrm{d}\omega) = 0 , \tag{21}$$

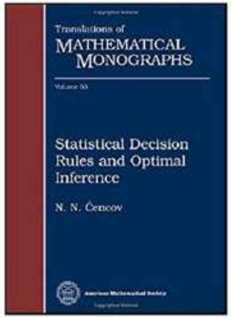
whatever $P_{s'}, P_{s''} \in \{P_s\}$ are, then

$$I(R: P_s) = I(R: P_\sigma) + I(P_\sigma: P_s), \quad \forall P_s . \tag{22}$$

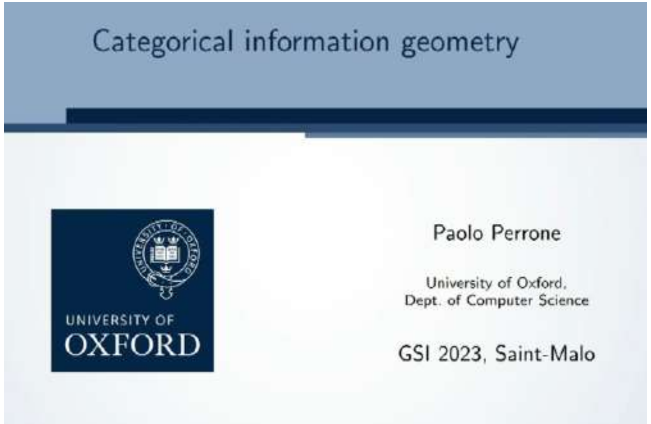
Corollary. *The measure P_σ is the I-nearest to R point of the exponential family.*

Recently a number of proposals for the arising non-symmetric Pyphagoric information geometry has been recovered by Csiszar [11].

Nikolai Nikolaevich Chentsov (1930-1992)



Category Theory & Information Geometry: Markov Category (GSI'23)



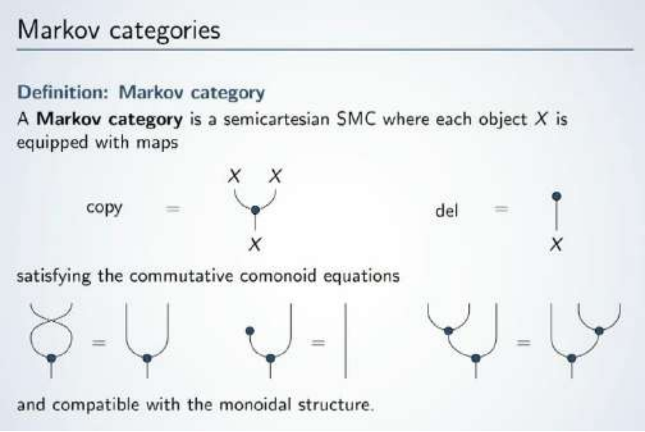
Categorical information geometry

Paolo Perrone

University of Oxford,
Dept. of Computer Science

GSI 2023, Saint-Malo

https://www.youtube.com/watch?v=Q97PrJ6ZpnI



Markov categories

Definition: Markov category

A **Markov category** is a semicartesian SMC where each object X is equipped with maps

copy =

del =

satisfying the commutative comonoid equations

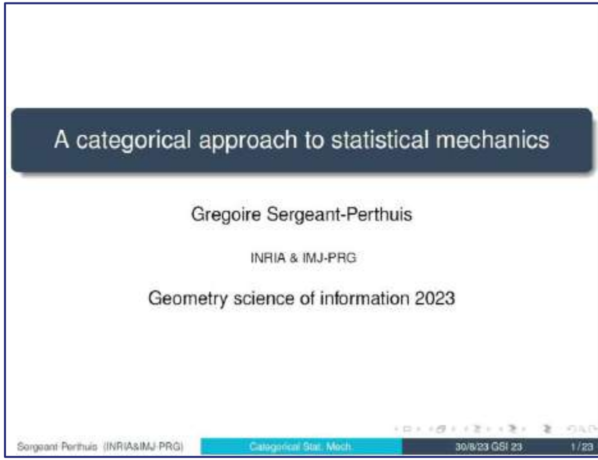
and compatible with the monoidal structure.

A TOPOS IS A PARTICULAR CASE OF CATEGORY (the category of sheaves of sets on a topological space)

Information geometry is the study of interactions between random variables by means of metric, divergences, and their geometry.

Categorical probability has a similar aim, but uses algebraic structures, primarily **monoidal categories**, for that purpose.

As recent work shows, we can **unify the two approaches by means of enriched category theory** into a single formalism, and recover important information-theoretic quantities and results, such as entropy and data processing inequalities



A categorical approach to statistical mechanics

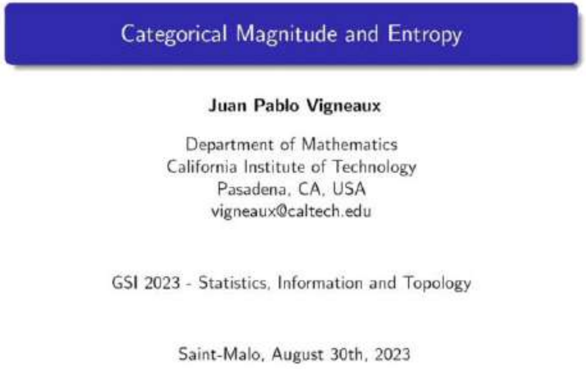
Gregoire Sergeant-Perthuis

INRIA & IMJ-PRG

Geometry science of information 2023

Sergeant-Perthuis (INRIA&IMJ-PRG) Categorical Stat. Mech. 30/8/23 GSI 23 1/23

https://www.youtube.com/watch?v=-z0583ynZoQ



Categorical Magnitude and Entropy

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Pasadena, CA, USA
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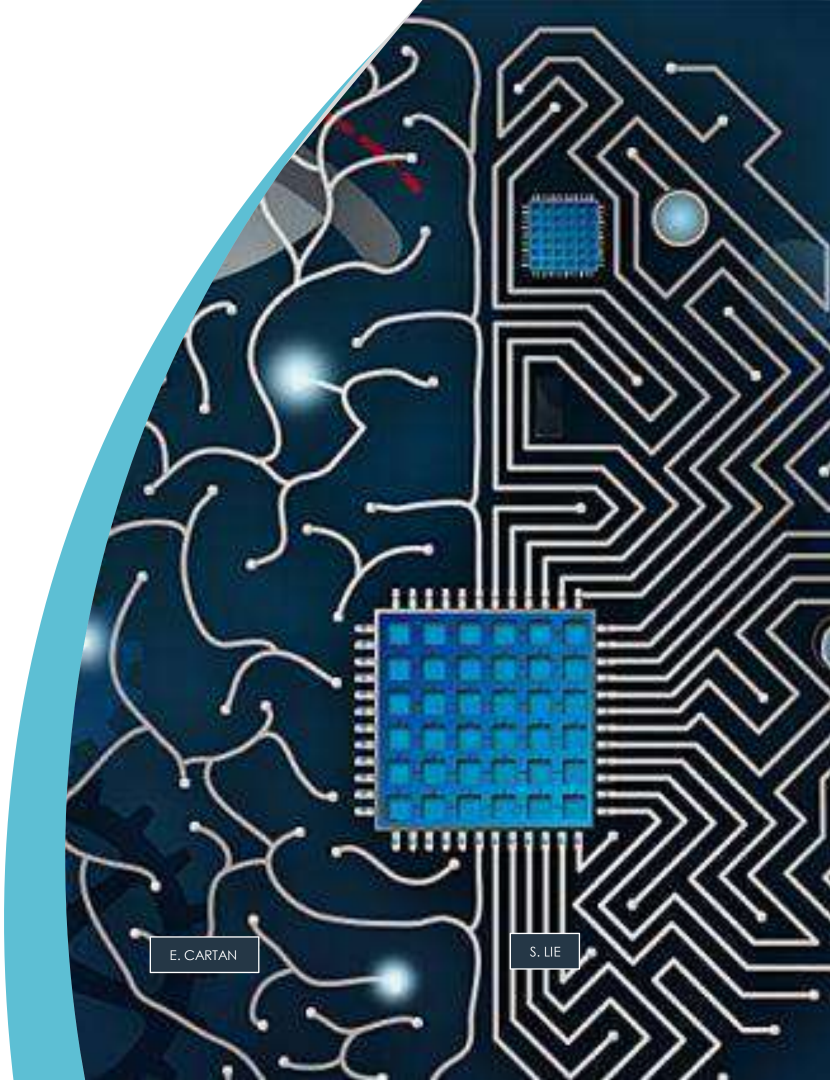
GSI 2023 - Statistics, Information and Topology

Saint-Malo, August 30th, 2023

tps://www.youtube.com/watch?v=1EC23Q6wfpw

Seminal Idea of Gibbs and Carathéodory



Carathéodory's axiomatization of Thermodynamics

[…] In any arbitrary neighborhood of an arbitrarily given initial point there is a state that cannot be arbitrarily approximated by adiabatic changes of state. (Carathéodory 1909)

- Constantin Carathéodory was born in Germany in Berlin in 1873 but was the son of a Turkish ambassador of Greek origin. From 1875 to 1895, Constantin Carathéodory lived in Bruxelles in Belgium, and studied at Belgian École Militaire. After a period in Greece, in London and Egypt, he returns to Berlin in 1900 to study mathematics.

- In 1904 in Göttingen, he defended his PhD on special Euler–Lagrange equations, with H. Minkowski,. He was successively professor in Göttingen, Bonn and Hannover. From 1905 to 1910. From Brussels, Carathéodory wrote to Born in 1907 about his definition of the concepts "amount of heat" and "reversibility" derived from the Carnot principle. He was back in Munich in 1924 to work on variational calculus applied to optics

Carathéodory's axiomatization of Thermodynamics

The main Axiom of Constantin Carathéodory of the 2nd law of Thermodynamics is written in 1909 paper:

[…] In jeder beliebigen Umgebung eins vorgeschriebenen Anfangszustandes gibt es Suztände, die durh adiabatische Zustandsänderungen nicht beliebig approximiert warden können [In the neighborhood of any equilibrium state of a system (of any number of thermodynamic coordinates), there exists states that are inaccessible by reversible adiabatic processes]. (Carathéodory 1909)

By analyzing the Pfaffian form, Carathéodory showed that if no heat exchange occurs, a system's adiabatic paths can be described mathematically, and implies the existence of an integrating factor related to temperature, leading to the definition of entropy. The integrating factor transforms the Pfaffian form into an exact differential, dS, which defines the change in entropy.

- **For the first law**, under any adiabatic change of state, the external work A by the change in energy is equal to zero and if we denote the initial ε and final $\bar{\varepsilon}$ energy:

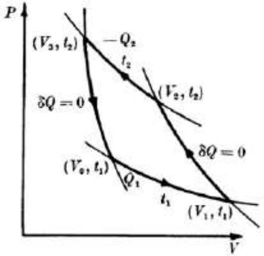
$$\bar{\varepsilon} - \varepsilon + A = 0$$

- **For the second law**, under all adiabatic changes of state that start from any given initial state, certain final states are not attainable and "unattainable" final states can be found in any neighborhood of the initial state.

Wolfgang Pauli Lectures on “Thermodynamics and the Kinetic Theory of Gases” – Rereading of Carathéodoty Axiomatization

- Heat conduction and Internal Friction are two main causes of dissipation
- […] If the second law were untrue, that is, if heat conduction were reversible, then work could be obtained from heat without compensating changes. Since it is only possible to convert work into heat without compensating changes, is the preferred direction in which actual processes take place. We can now also determine which of two temperature t1 and t2 is the higher, namely, that one from which heat can flow to the other without compensating changes.

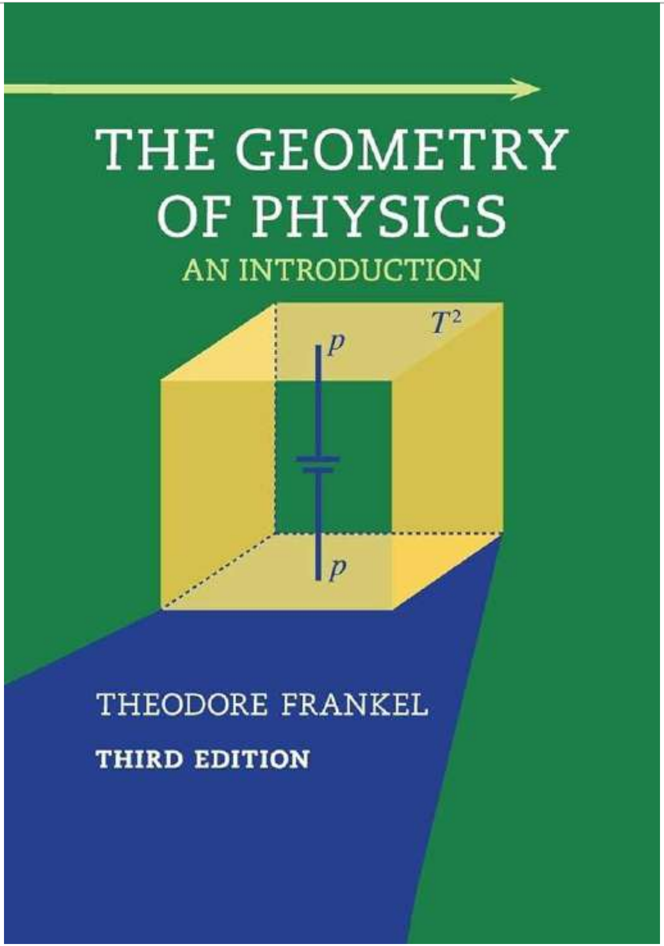
(Pauli 1973)



Wolfgang Pauli Lectures on Physics, Volume 3 “Thermodynamics and the Kinetic Theory of Gases”, 2nd chapter dedicated to the 2nd Law:

[…] The second law distinguishes heat from the other forms of energy. It indicates a direction in time and makes apparent that heat is a disordered form of energy. (Pauli 1973)

Constantin Carathéodory axiomatization of thermodynamics interpreted by foliations by Theodore Frankel



[…] If, for example, **the foliation defined by $Q = 0$ has leaves that wind densely (as in a torus) then there is no way that a global function S can exist, since such an S must be constant on each maximal leaf.** It is easy to see, however, that Kelvin's second law of thermodynamics rules out the possibility of not only dense adiabatic leaves, but even leaves that "double back"! For in the proof that "Kelvin implies Caratheodory," we saw that two states related by heating at constant volume cannot be joined by a quasi-static adiabatic. This says that no $\pi^{-1}(v)$ can meet a maximal adiabatic leaf twice. It might be thought that the space M^{n+1} and the adiabatic foliation must then be of a completely trivial nature. **(Frankel 1972, p.33)**

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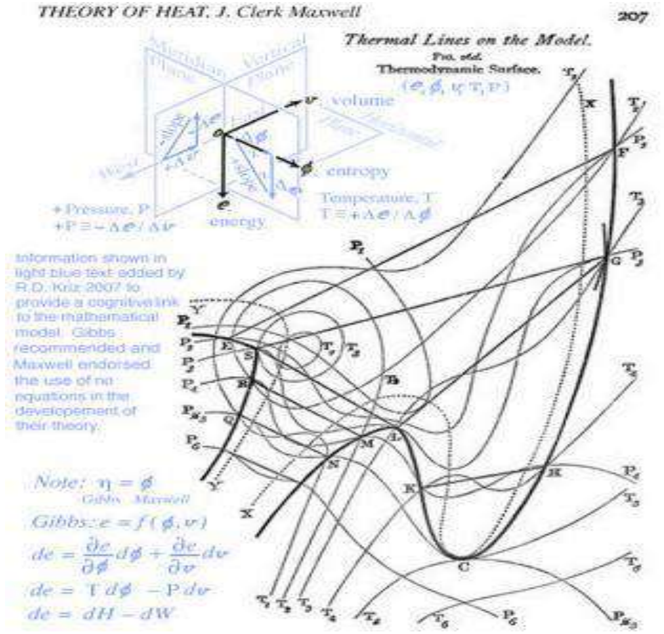
Ryszard Mrugala : Information Geometry and Libermann Legendre submanifolds

- We make reference to Ryszard Mrugala paper on “**On contact and metric structures on thermodynamic spaces**” (Mrugala 1978 & 2000), introducing geometrical structures of the state space of classical thermodynamics:

  - contact structure for the first law of thermodynamics
  - metrical Riemannian structure for the second law of thermodynamics.

- In this paper, R. Mrugala considered contact transformations and thermodynamic symmetries, more especially **a group of diffeomorphisms (symmetries) which preserve its contact structure.** He developed a 1-parameter group of continuous contact transformation, which preserves the contact distribution. He showed that contact vector field is a generator of a continuous contact transformation and that the fields of contact vectors form a Lie algebra. **He observed that the contact Hamiltonian flow in the general case is tangent only to one level surface.** He finally showed that using a generalized canonical Gibbs probability distribution function (Jaynes 1957) and relaxing some standard conditions, we can introduce a contact and a metric structure on the space of thermodynamic parameters. Finally, Riemannian metric is introduce on **Legendre submanifolds, where the metric tensor is equivalent to Fisher’s information matrix from Information Geometry.**

Geometric Theory of Heat: Josiah Willard Gibbs Diagrams sculpted by James Clerk Maxwell (1874)


THEORY OF HEAT. J. Clerk Maxwell 207

Thermal Lines on the Model.

Thermodynamic Surface.

Information shown in light blue text added by R.D. Kriz 2007 to provide a cognitive link to the mathematical model. Gibbs recommended and Maxwell endorsed the use of no equations in the developement of their theory.

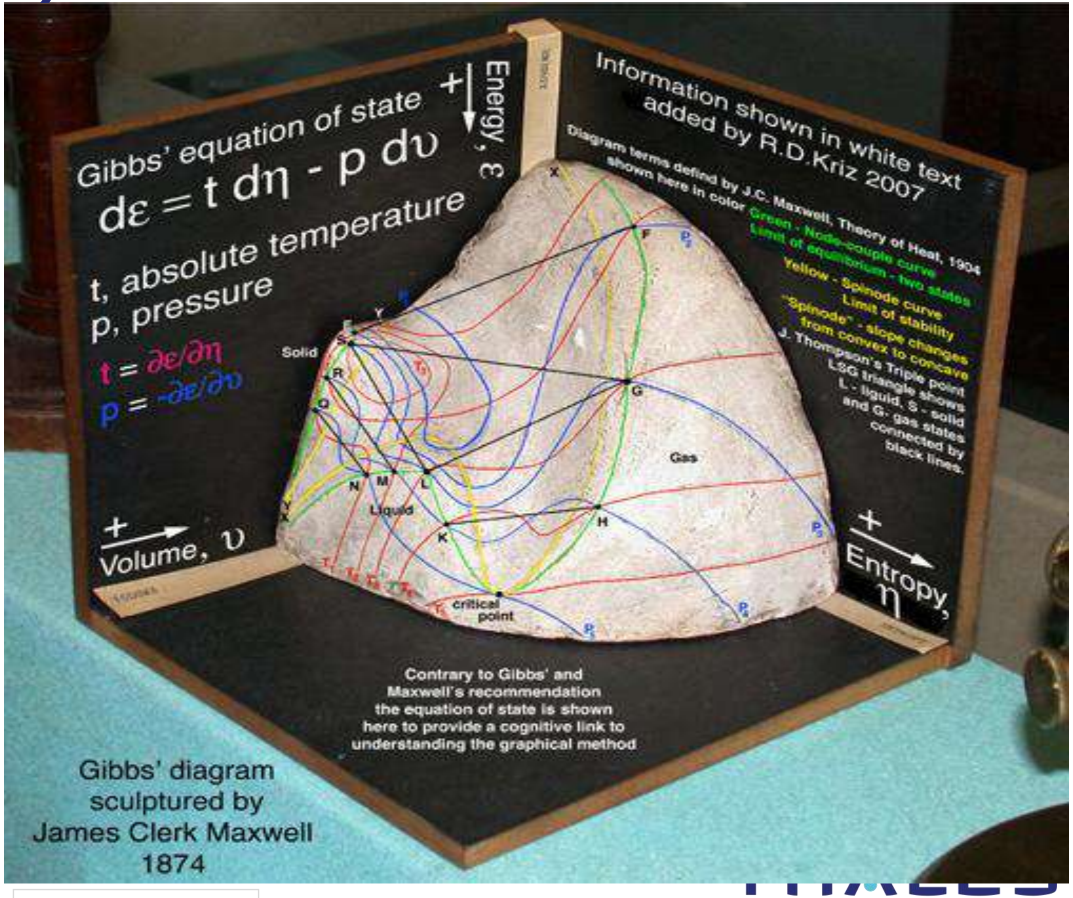
Note: $\eta = \phi$ (Gibbs, Maxwell)

Gibbs: $e = f(\phi, v)$

$$de = \frac{\partial e}{\partial \phi} d\phi + \frac{\partial e}{\partial v} dv$$

$$de = T\,d\phi - P\,dv$$

$$de = dH - dW$$


Gibbs' equation of state

$$d\varepsilon = t\,d\eta - p\,dv$$

t, absolute temperature

p, pressure

$$t = \partial\varepsilon/\partial\eta$$

$$p = -\partial\varepsilon/\partial v$$

Information shown in white text added by R.D.Kriz 2007

Diagram terms defind by J.C. Maxwell, Theory of Heat, 1904 shown here in color Green - Node-couple curve Limit of equilibrium - two states Yellow - Spinode curve Limit of stability "Spinode" - slope changes from convex to concave J. Thompson's Triple point LSG triangle shows L - liquid, S - solid and G - gas states connected by black lines.

Contrary to Gibbs' and Maxwell's recommendation the equation of state is shown here to provide a cognitive link to understanding the graphical method

Gibbs' diagram sculptured by James Clerk Maxwell 1874

Seminal Gibbs States from Moment Map of SE(3)

- As observed by Charles-Michel Marle, **Josiah Willard Gibbs** in chapter IV of his book “Elementary Principles in Statistical Mechanics, developed with Especial Reference to the Rational Foundation of Thermodynamics” published in 1902, considered **generalization of Gibbs states built with the moment map of the product of the one-dimensional group of translations in time and the three-dimensional group of rotations in space for the study of systems contained in a rotating vessel**, referring to a paper by Maxwell published in 1878. We can read in the book of Gibbs:

[…] The consideration of the above case of statistical equilibrium may be made the foundation of the theory of the thermodynamic equilibrium of rotating bodies, a subject which has been treated by Maxwell in his memoir On Boltzmann's theorem on the average distribution of energy in a system of material points (Gibbs 1902)

- **Jacques Hadamard** made a review of this book in 1906 (Hadamard 1906) and wrote:

[…] This book is not one of those that one analyzes hastily; but, on the other hand, the questions it deals with have been greatly agitated in recent times; the ideas defended by Gibbs have been the subject of much controversy; the reasoning with which he supported them has also been criticized. It seems interesting to me to study his work in the light of these controversies and by discussing these criticisms (Hadamard 1906)

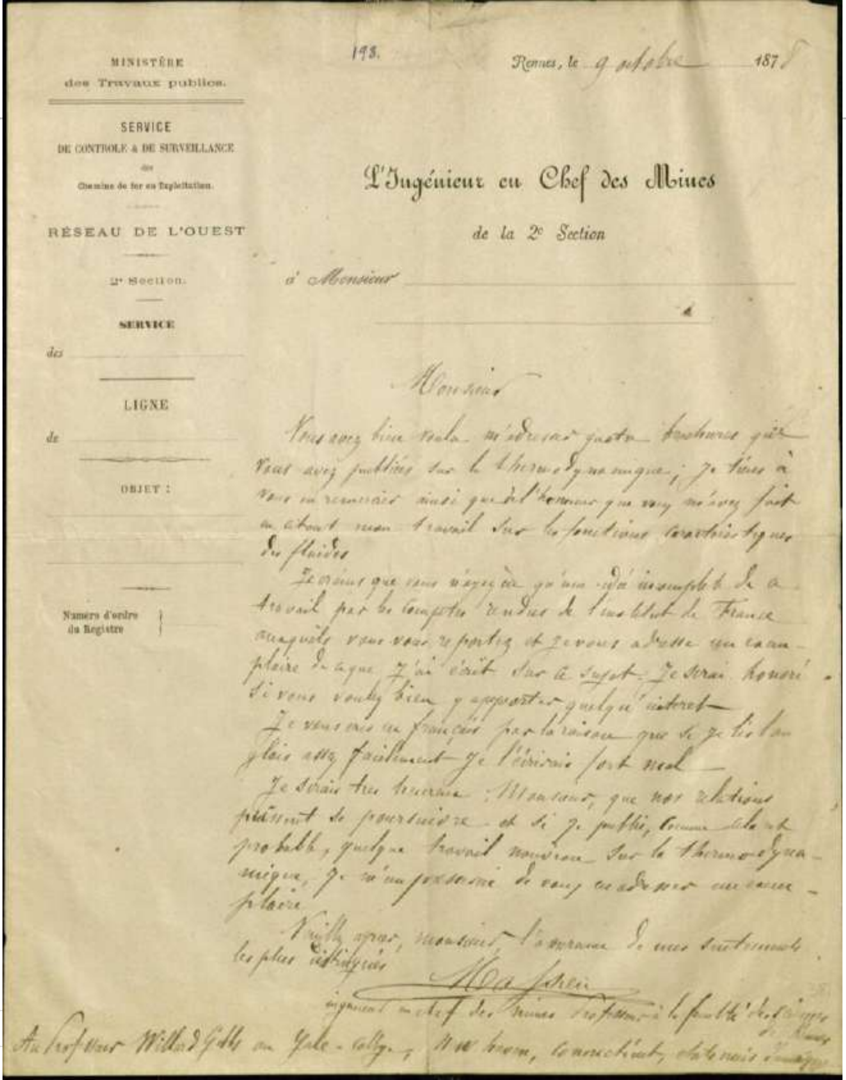
GIBBS, MASSIEU & THERMODYNAMICAL POTENTIALS

- **Handwritten letter from François Massieu (Corps des Mines) to Josiah Willard Gibbs, dated Rennes, October 9, 1878 (Gibbs Archive of Yale University):**
- "You have been kind enough to send me four brochures that you have published on thermodynamics; I would like to thank you as well as for the honor you have done me by citing my work on the characteristic functions of fluids. I fear that you have had only an incomplete idea of this work for the reports of the Institut de France to which you refer and I am sending you a copy of what I have written on this subject. I would be honored if you would be kind enough to bring some interest to it. I am writing to you in French for the reason that if I read English quite easily, I would write it very badly. I would be very happy, Sir, if our relations could continue and if I publish, as is likely, some new work on thermodynamics, I will hasten to send you a copy. Please accept, Sir, the assurance of my most distinguished sentiments.

MASSIEU, Chief Engineer of Mines

Professor at the Faculty of Sciences of Rennes

to Professor Willard Gibbs, at Yale College, New Haven, Connecticut, United States of America"



MINISTÈRE
des Travaux publics.

SERVICE
DE CONTROLE & DE SURVEILLANCE
des
Chemins de fer en Exploitation.

RÉSEAU DE L'OUEST

2e Section.

SERVICE
des

LIGNE
de

OBJET :

Numéro d'ordre
du Registre

193.

Rennes, le 9 octobre 1878

L'Ingénieur en Chef des Mines
de la 2e Section

à Monsieur

Monsieur

Vous avez bien voulu m'adresser quatre brochures que vous avez publiées sur la thermodynamique; je tiens à vous en remercier ainsi que de l'honneur que vous m'avez fait en citant mon travail sur les fonctions caractéristiques des fluides.

Je crains que vous n'ayez eu qu'une idée incomplète de ce travail par les comptes rendus de l'Institut de France auxquels vous vous reportez et je vous adresse un exemplaire de ce que j'ai écrit sur ce sujet. Je serai honoré si vous voulez bien y apporter quelque intérêt.

Je vous écris en français par la raison que si je lis l'anglais assez facilement je l'écrirais fort mal.

Je serais très heureux, Monsieur, que nos relations puissent se poursuivre et si je publie, comme cela est probable, quelque travail nouveau sur la thermodynamique, je m'empresserai de vous en adresser un exemplaire.

Veuillez agréer, Monsieur, l'assurance de mes sentiments les plus distingués.

Massieu

ingénieur en chef des mines professeur à la faculté des sciences de Rennes

Au Professeur Willard Gibbs au Yale-College, New Haven, Connecticut, États-Unis d'Amérique

Thermodynamics as a Science of Symmetry by Herbert B. Callen

- […] every continuous symmetry of a system implies a conservation theorem, and vice versa … The most primitive class of symmetries is the class of continuous spacetime transformations. The (presumed) invariance of physical laws under time translation implies the conservation of energy. **Symmetry under spatial translation implies conservation of momentum, and rotational symmetry implies conservation of angular momentum**.(Callen, p.425)

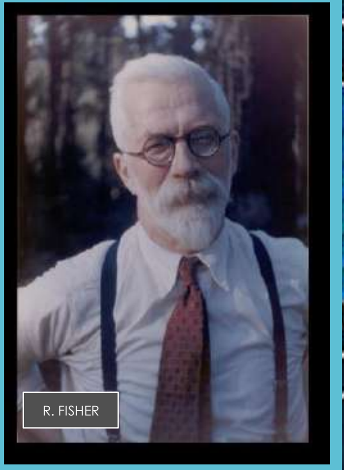
- […] The most immediately evident conserved coordinate is, of course, the energy (time-translation symmetry). **Its relevance as a thermodynamic coordinate under-lies the "first law" of thermodynamics. Time-translation, spatial translation, and spatial rotation symmetries are interrelated in a single class of continuous space-time symmetries.** The symmetry interpretation of thermodynamics immediately suggests, then, that **energy, linear momentum, and angular momentum should play fully analogous roles in thermodynamics.** The equivalence of these roles is rarely evident in conventional treatments, which appear to grant the energy a mis-leadingly unique status. **The momentum and the angular momentum are generally suppressed by restricting the theory to systems at rest, constrained by external "clamps."** Nevertheless, it is evident that in principle **the linear momentum does appear in the formalism in a form fully equivalent to the energy**, for relativistic considerations imply that the energy in one frame appears partially as linear momentum in another frame. Similarly, the angular momentum is only occasionally introduced explicitly into thermodynamic formalisms (as in astrophysical applications to rotating galaxies); it appears, for instance, in the "Boltzmann factor," $\exp\left(-\beta E-\beta\Lambda.L\right)$, additively and symmetrically with the energy. To stress these facts we might well amend the first law to read that "**the extended first law of thermodynamics is the symmetry of the laws of physics under space and time translations and under spatial rotation**.".(Callen 1974 p.427)


Herbert B. Callen

- **Callen HB (1973) A Symmetry Interpretation of Thermodynamics. In: Domingos JDD (eds.). Foundations of Continuum Thermodynamics, Instituto de Alta Cultura-Núcleo de Estudos de Engenharia Mecanica, pp. 61–79.**
- **Callen, HB (1974) Thermodynamics as a Science of Symmetry, Foundations of Physics 4(4):423–443.**
- **Callen HB (1985) Thermodynamics and An Introduction to Thermostatistics, 2nd edition, Wiley, New York.**

THALES
Building a future we can all trust

Information Geometry & Natural Gradient for Learning

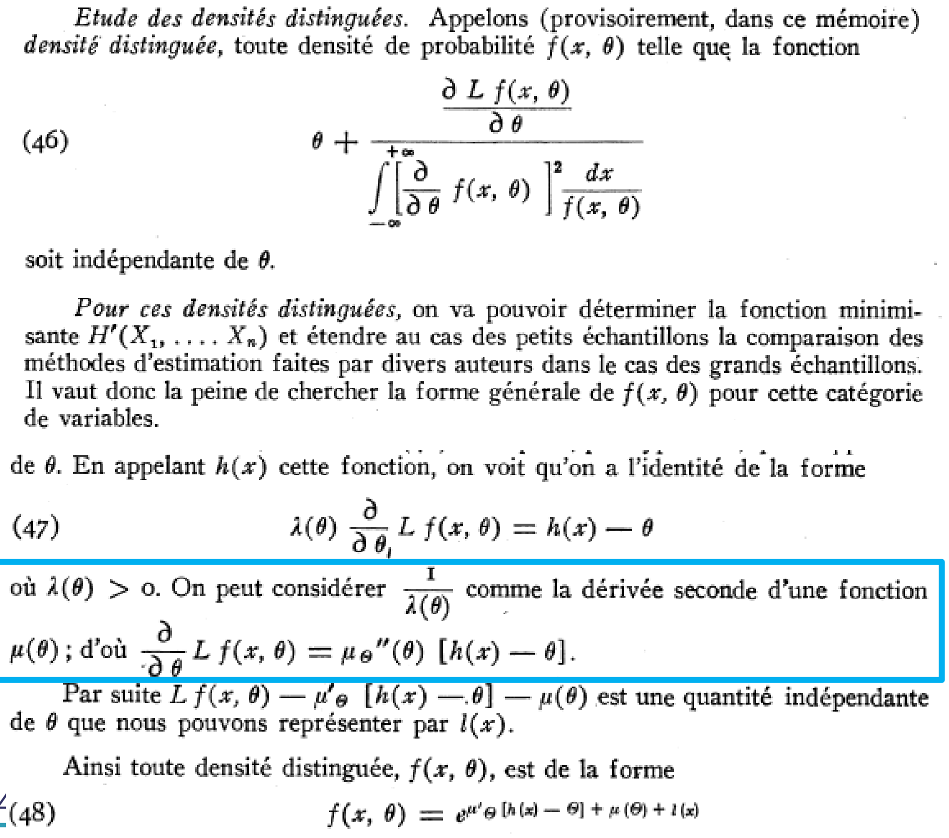


R. FISHER

N.N. CHENTSOV

Maurice Fréchet 1943 Seminal Paper (Clairaut Equation)

Fréchet, M. Sur l'extension de certaines évaluations statistiques au cas de petits échantillons. Revue de l'Institut International de Statistique 1943, 11, 182–205.



Etude des densités distinguées. Appelons (provisoirement, dans ce mémoire) *densité distinguée*, toute densité de probabilité $f(x, \theta)$ telle que la fonction

$$\theta + \frac{\dfrac{\partial L f(x, \theta)}{\partial \theta}}{\displaystyle\int_{-\infty}^{+\infty} \left[\frac{\partial}{\partial \theta} f(x, \theta)\right]^2 \frac{dx}{f(x, \theta)}} \tag{46}$$

soit indépendante de θ.

Pour ces densités distinguées, on va pouvoir déterminer la fonction minimisante $H'(X_1, \ldots . X_n)$ et étendre au cas des petits échantillons la comparaison des méthodes d'estimation faites par divers auteurs dans le cas des grands échantillons. Il vaut donc la peine de chercher la forme générale de $f(x, \theta)$ pour cette catégorie de variables.

de θ. En appelant $h(x)$ cette fonction, on voit qu'on a l'identité de la forme

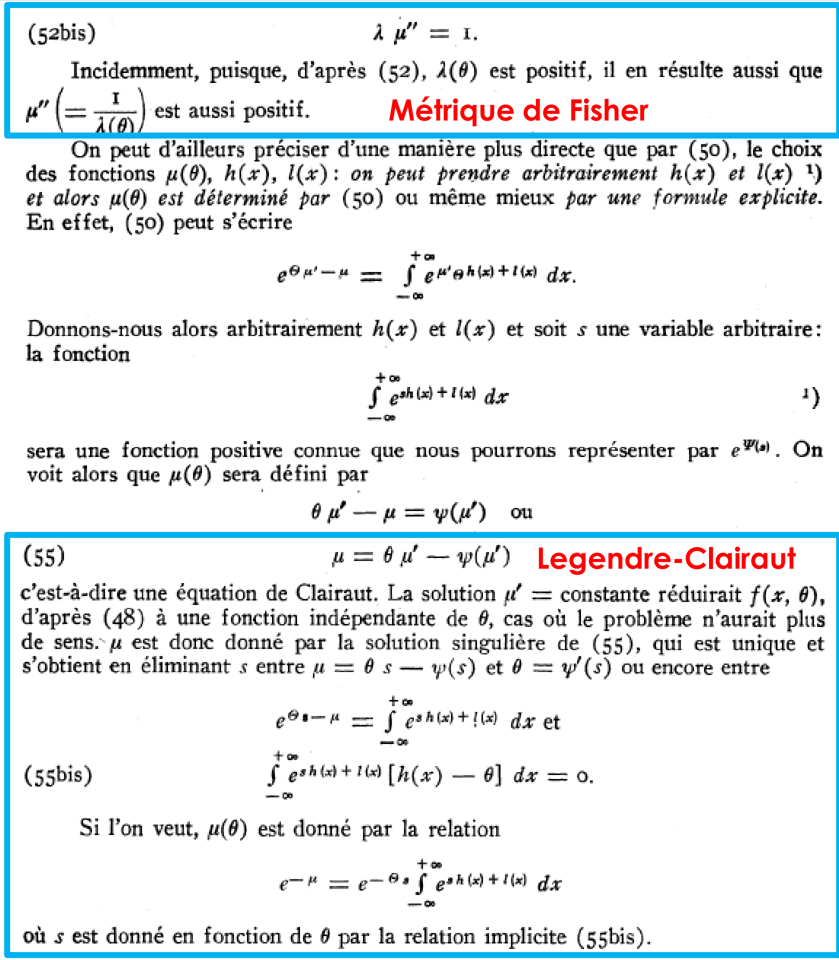
$$\lambda(\theta) \frac{\partial}{\partial \theta} L f(x, \theta) = h(x) - \theta \tag{47}$$

où $\lambda(\theta) > 0$. On peut considérer $\frac{1}{\lambda(\theta)}$ comme la dérivée seconde d'une fonction $\mu(\theta)$; d'où $\frac{\partial}{\partial \theta} L f(x, \theta) = \mu_\Theta''(\theta) \, [h(x) - \theta]$.

Par suite $L f(x, \theta) - \mu'_\Theta \, [h(x) - \theta] - \mu(\theta)$ est une quantité indépendante de θ que nous pouvons représenter par $l(x)$.

Ainsi toute densité distinguée, $f(x, \theta)$, est de la forme

$$f(x, \theta) = e^{\mu'_\Theta [h(x) - \Theta] + \mu(\Theta) + l(x)} \tag{48}$$



$$\lambda \mu'' = 1. \tag{52bis}$$

Incidemment, puisque, d'après (52), $\lambda(\theta)$ est positif, il en résulte aussi que $\mu'' \left(= \frac{1}{\lambda(\theta)}\right)$ est aussi positif.

Métrique de Fisher

On peut d'ailleurs préciser d'une manière plus directe que par (50), le choix des fonctions $\mu(\theta)$, $h(x)$, $l(x)$: *on peut prendre arbitrairement $h(x)$ et $l(x)$* [1]) *et alors $\mu(\theta)$ est déterminé par* (50) ou même mieux *par une formule explicite.* En effet, (50) peut s'écrire

$$e^{\Theta \mu' - \mu} = \int_{-\infty}^{+\infty} e^{\mu'_\Theta h(x) + l(x)} \, dx.$$

Donnons-nous alors arbitrairement $h(x)$ et $l(x)$ et soit s une variable arbitraire : la fonction

$$\int_{-\infty}^{+\infty} e^{s h(x) + l(x)} \, dx \qquad {}^{1})$$

sera une fonction positive connue que nous pourrons représenter par $e^{\Psi(s)}$. On voit alors que $\mu(\theta)$ sera défini par

$$\theta \mu' - \mu = \psi(\mu') \quad \text{ou}$$

$$\mu = \theta \mu' - \psi(\mu') \tag{55}$$

Legendre-Clairaut

c'est-à-dire une équation de Clairaut. La solution μ' = constante réduirait $f(x, \theta)$, d'après (48) à une fonction indépendante de θ, cas où le problème n'aurait plus de sens. μ est donc donné par la solution singulière de (55), qui est unique et s'obtient en éliminant s entre $\mu = \theta s - \psi(s)$ et $\theta = \psi'(s)$ ou encore entre

$$e^{\Theta s - \mu} = \int_{-\infty}^{+\infty} e^{s h(x) + l(x)} \, dx \text{ et}$$

$$\int_{-\infty}^{+\infty} e^{s h(x) + l(x)} [h(x) - \theta] \, dx = 0. \tag{55bis}$$

Si l'on veut, $\mu(\theta)$ est donné par la relation

$$e^{-\mu} = e^{-\Theta s} \int_{-\infty}^{+\infty} e^{s h(x) + l(x)} \, dx$$

où s est donné en fonction de θ par la relation implicite (55bis).

Fisher Metric and Fréchet-Darmois (Cramer-Rao) Bound

- Cramer-Rao –Fréchet-Darmois Bound has been introduced by Fréchet in 1939 and by Rao in 1945 as inverse of the Fisher Information Matrix: $I(\theta)$

$$R_{\hat{\theta}} = E\left[\left(\theta - \hat{\theta}\right)\left(\theta - \hat{\theta}\right)^{+}\right] \geq I(\theta)^{-1} \qquad \left[I(\theta)\right]_{i,j} = -E\left[\frac{\partial^2 \log p_\theta(z)}{\partial\theta_i \partial\theta_j^*}\right]$$

- Rao has proposed to introduced an invariant metric in parameter space of density of probabilities (axiomatised by N. Chentsov):

$$ds_\theta^2 = Kullback\_Divergence\left(p_\theta(z), p_{\theta+d\theta}(z)\right)$$

$$ds_\theta^2 = -\int p_\theta(z) \log \frac{p_{\theta+d\theta}(z)}{p_\theta(z)} dz$$

$$w = W(\theta) \Rightarrow ds_w^2 = ds_\theta^2$$

$$ds_\theta^2 \underset{Taylor}{\approx} \sum_{i,j} g_{ij} d\theta_i d\theta_j^* = \sum_{i,j} \left[I(\theta)\right]_{i,j} d\theta_i d\theta_j^* = d\theta^+ . I(\theta) . d\theta$$

Distance Between Gaussian Density with Fisher Metric

Fisher Matrix for Gaussian Densities:

$$I(\theta)=\begin{bmatrix}\frac{1}{\sigma^2} & 0\\ 0 & \frac{2}{\sigma^2}\end{bmatrix} \quad \text{avec} \quad E\left[\left(\theta-\hat{\theta}\right)\left(\theta-\hat{\theta}\right)^T\right]\geq I(\theta)^{-1} \quad \text{et} \quad \theta=\begin{pmatrix}m\\ \sigma\end{pmatrix}$$

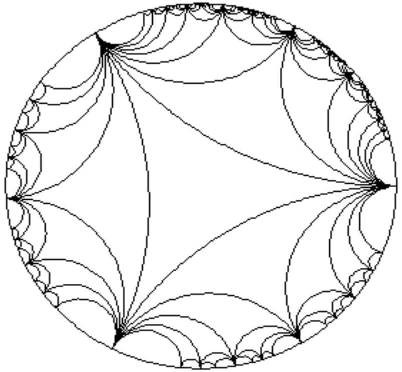
- Fisher matrix induced the following differential metric :

$$ds^2=d\theta^T.I(\theta).d\theta=\frac{dm^2}{\sigma^2}+2.\frac{d\sigma^2}{\sigma^2}=\frac{2}{\sigma^2}\left[\left(\frac{dm}{\sqrt{2}}\right)^2+(d\sigma)^2\right]$$

- Poincaré Model of upper half-plane and unit disk

$$z=\frac{m}{\sqrt{2}}+i.\sigma \qquad \omega=\frac{z-i}{z+i} \quad (|\omega|<1)$$

$$\Rightarrow ds^2=8.\frac{|d\omega|^2}{\left(1-|\omega|^2\right)^2}$$



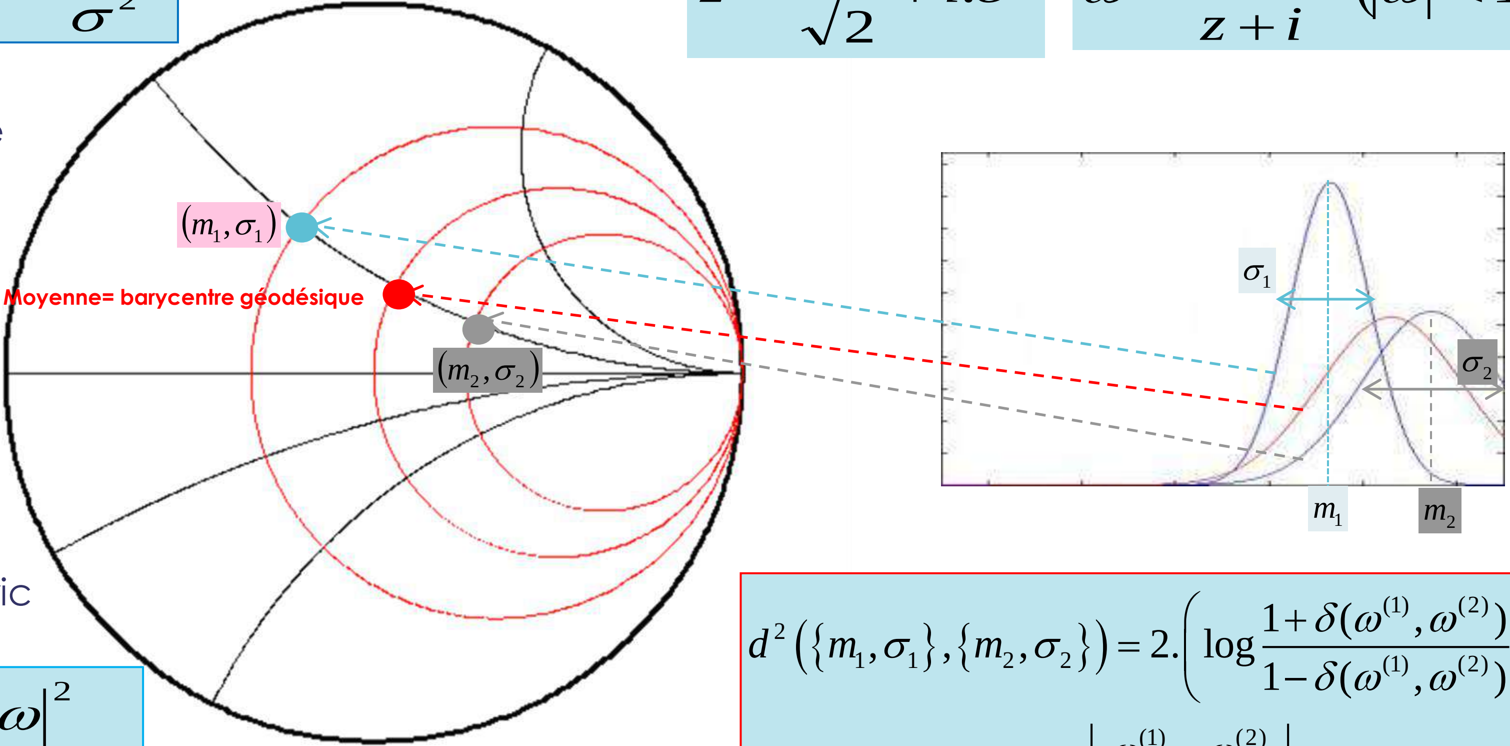
1 monovariate gaussian = 1 point in Poincaré unit disk

$$ds^2 = \frac{dm^2}{\sigma^2} + 2.\frac{d\sigma^2}{\sigma^2}$$

Fisher Metric in
Poincaré Half-Plane

$$z = \frac{m}{\sqrt{2}} + i.\sigma$$

$$\omega = \frac{z-i}{z+i} \quad \left(|\omega| < 1\right)$$



Poincaré-Fisher metric
In Unit Disk

$$ds^2 = 8.\frac{|d\omega|^2}{\left(1-|\omega|^2\right)^2}$$

$$d^2\left(\{m_1,\sigma_1\},\{m_2,\sigma_2\}\right) = 2.\left(\log\frac{1+\delta(\omega^{(1)},\omega^{(2)})}{1-\delta(\omega^{(1)},\omega^{(2)})}\right)^2$$

$$\text{with} \quad \delta(\omega^{(1)},\omega^{(2)}) = \left|\frac{\omega^{(1)}-\omega^{(2)}}{1-\omega^{(1)}\omega^{(2)*}}\right|$$

Information Geometry & Machine Learning

Information Geometry & Natural Gradient

- This simple gradient descent has a first drawback of using the same non-adaptive learning rate for all parameter components, and a second drawback of non invariance with respect to parameter re-encoding inducing different learning rates. **S.I. Amari** has introduced the **natural gradient** to preserve this invariance to be insensitive to the characteristic scale of each parameter direction. The gradient descent could be corrected by $I(\theta)^{-1}$ where I is the **Fisher information matrix** with respect to parameter θ , given by:

$$\theta_t \leftarrow \theta_{t-1} - \eta_t I(\theta_{t-1})^{-1} \frac{\partial l_t(y_t)^T}{\partial \theta}$$

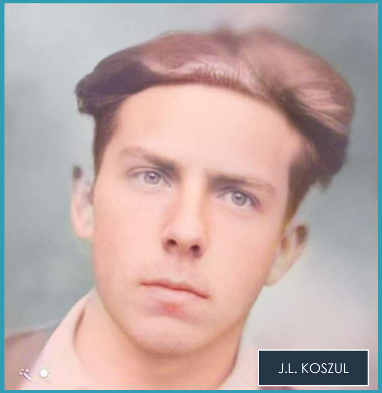
$$I(\theta) = \left[g_{ij} \right]$$

$$\text{with } g_{ij} = \left[-E_{y\approx p(y/\theta)} \left[\frac{\partial^2 \log p(y/\theta)}{\partial \theta_i \partial \theta_j} \right] \right]_{ij} = \left[E_{y\approx p(y/\theta)} \left[\frac{\partial \log p(y/\theta)}{\partial \theta_i} \frac{\partial \log p(y/\theta)}{\partial \theta_j} \right] \right]_{ij}$$

THALES

Building a future we can all trust

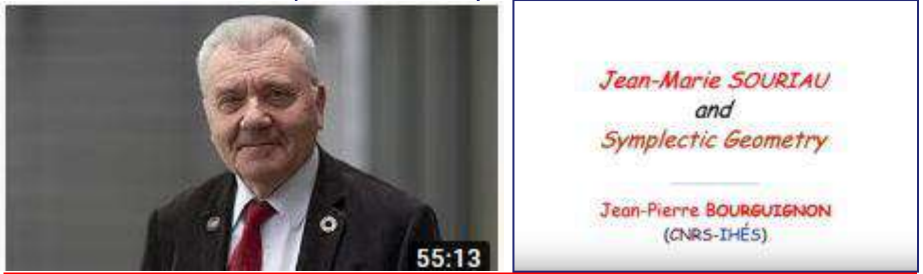
Jean-Marie Souriau:
Symplectic Foliation via Moment Map and Koszul-Fisher Metric from Information Geometry for Lie Groups Thermodynamics



SOURIAU 2019

SOURIAU 2019

- Internet website : http://souriau2019.fr
- In 1969, 50 years ago, Jean-Marie Souriau published the book "**Structure des système dynamiques**", in which using the ideas of J.L. Lagrange, he formalized the "**Geometric Mechanics**" in its modern form based on **Symplectic Geometry**
- Chapter IV was dedicated to "Thermodynamics of Lie groups" (ref André Blanc-Lapierre)
- Testimony of **Jean-Pierre Bourguignon** at Souriau'19 (IHES director of the European ERC)



Jean-Marie SOURIAU
and
Symplectic Geometry

Jean-Pierre BOURGUIGNON
(CNRS-IHÉS)

https://www.youtube.com/watch?v=93hFollBo0Q&t=3s

SOURIAU 2019

Conference May 27-31 2019, Paris-Diderot University

https://www.youtube.com/watch?v=beM2pUK1H7o

Frédéric Barbaresco
Daniel Bennequin
Jean-Pierre Bourguignon
Pierre Cartier
Dan Christensen
Maurice Courbage
Thibault Damour
Paul Donato
Paolo Giordano
Serap Gürer
Patrick Iglesias-Zemmour
Yael Karshon
Jean-Pierre Magnot
Yvette Kosmann-Schwarzbach
Marc Lachièze-Rey
Martin Pinsonnault
Elisa Prato
Urs Schreiber
Jean-Jacques Szczeciniarz
Roland Triay
Jordan Watts
Enxin Wu
San Vũ Ngọc
Alan Weinstein

JEAN-MARIE SOURIAU

In 1969, the groundbreaking book of Jean-Marie Souriau appeared "Structure des Systèmes Dynamiques". We will celebrate, in 2019, the jubilee of its publication, with a conference in honour of the work of this great scientist.

Symplectic Mechanics, Geometric Quantization, Relativity, Thermodynamics, Cosmology, Diffeology & Philosophy

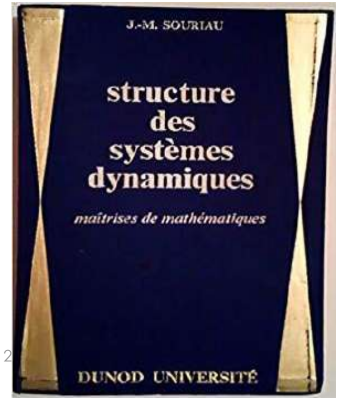
Machine Learning on Lie Groups

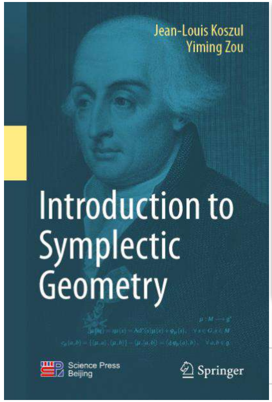
Lie Groups Machine Learning based on Souriau Lie-Groups Thermodynamics

- We will introduce "**Lie Groups Machine Learning**" [1] that extends statistics and machine learning on Lie Groups based on representation theory and Lie algebra cohomology.
- From Jean-Marie Souriau "**Lie Groups Thermodynamics**" [3] initiated in the framework of symplectic model of statistical mechanics, new geometric statistical tools have been developed to define:

  - maximum Entropy densities of probability on Lie Groups for supervised methods

  - Fisher metric extension from Information Geometry for unsupervised methods.



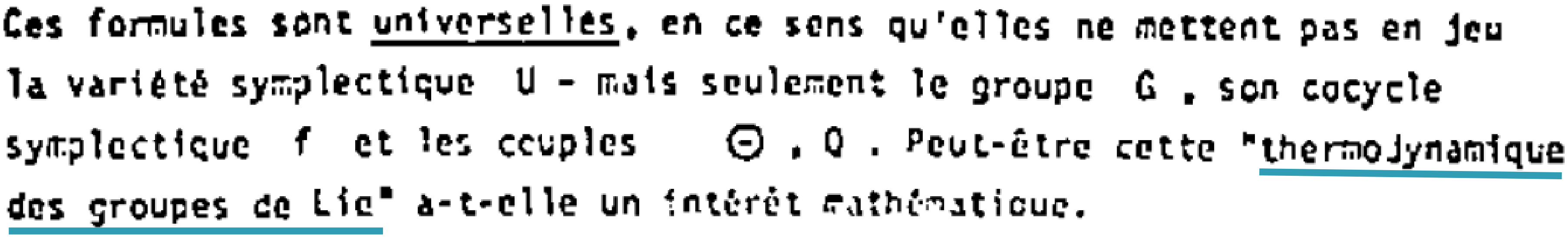


> *« Il n'y a rien de plus dans les théories physiques que les groupes de symétrie si ce n'est la construction mathématique qui permet précisément de montrer qu'il n'y a rien de plus »* - **Jean-Marie Souriau**

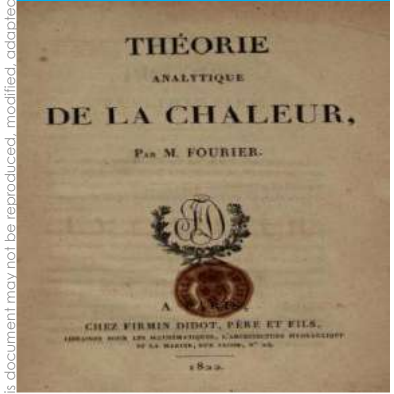
Souriau's Moment Map =
Geometrization of Noether Theorem

Building a future we can all trust

Souriau Lie Groups Thermodynamics: General Equations


Ces formules sont universelles, en ce sens qu'elles ne mettent pas en jeu la variété symplectique U - mais seulement le groupe G , son cocycle symplectique f et les couples Θ , Q . Peut-être cette "thermodynamique des groupes de Lie" a-t-elle un intérêt mathématique.

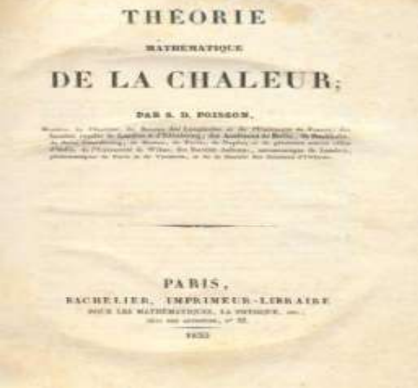
FOURIER
Analytical Theory of Heat


THÉORIE
ANALYTIQUE
DE LA CHALEUR,
PAR M. FOURIER.
A PARIS,
CHEZ FIRMIN DIDOT, PÈRE ET FILS,
1822.

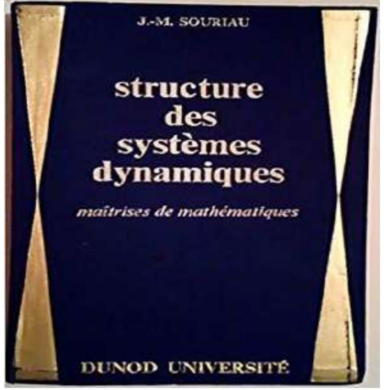
CLAUSIUS
Mechanical Theory of Heat


THÉORIE MÉCANIQUE
DE
LA CHALEUR
PAR
R. CLAUSIUS
PAR F. FOLIE
PARIS
LIBRAIRIE SCIENTIFIQUE, INDUSTRIELLE ET AGRICOLE
EUGÈNE LACROIX, ÉDITEUR

POISSON
Mathematical Theory of Heat


THÉORIE
MATHÉMATIQUE
DE LA CHALEUR;
PAR S. D. POISSON,
PARIS,
BACHELIER, IMPRIMEUR-LIBRAIRE

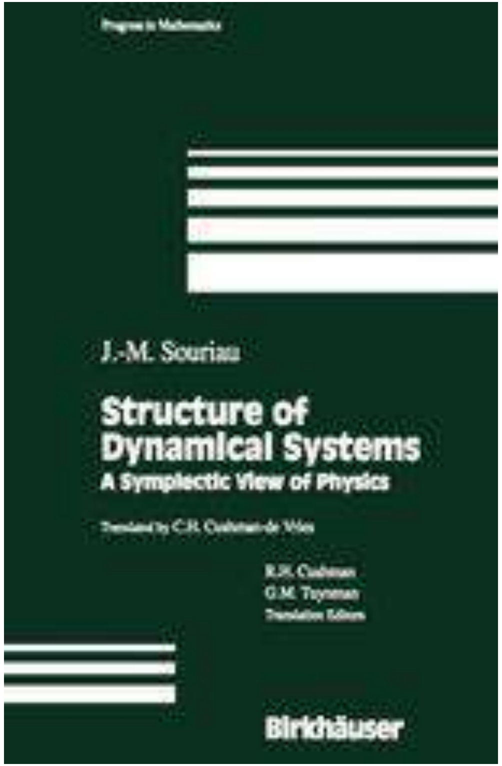
SOURIAU
Geometric Theory of Heat

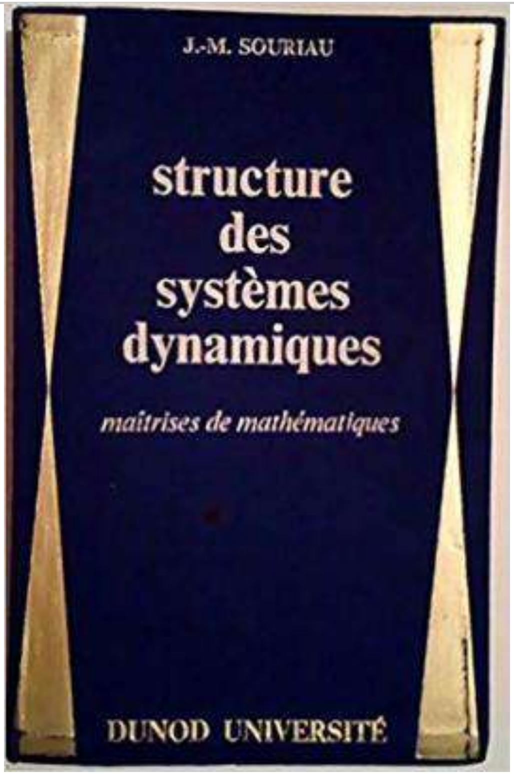

J.-M. SOURIAU
structure
des
systèmes
dynamiques
maîtrises de mathématiques
DUNOD UNIVERSITÉ

SYMPLECTIC THEORY OF HEAT

Building a future we can all trust

J.M. Souriau Book « Structure des systèmes dynamiques », 1969

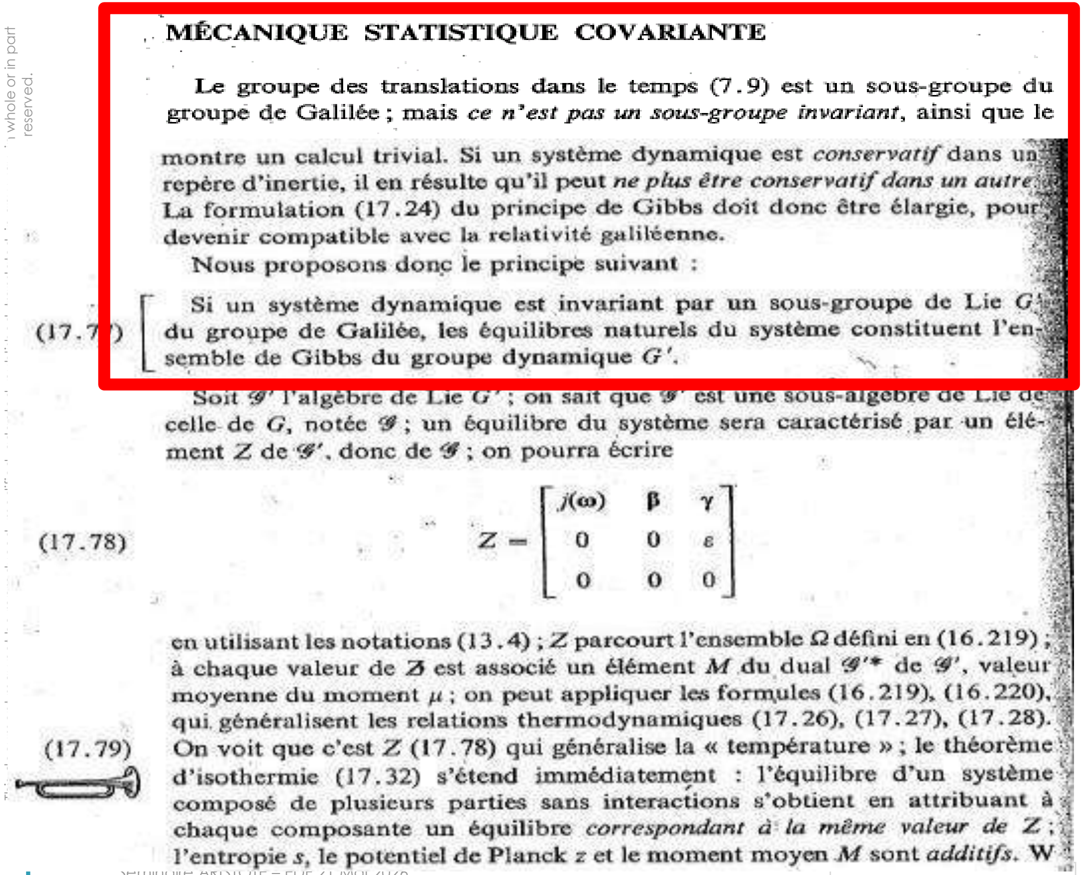




- Introduction of symplectic geometry in mechanics
- Invention of the "moment map"
- Geometrization of Noether's theorem
- Barycentric decomposition theorem
- The total mass of an isolated dynamic system is the class of cohomology of the default of equivariance for the moment map
- Lie Groups Thermodynamics (Chapter IV)

http://www.jmsouriau.com/structure_des_systemes_dynamiques.htm
http://www.springer.com/us/book/9780817636951

Souriau SSD Chapter IV: Gibbs Equilibrium is not covariant with respect to Dynamic Groups of Physics



MÉCANIQUE STATISTIQUE COVARIANTE

Le groupe des translations dans le temps (7.9) est un sous-groupe du groupe de Galilée ; mais *ce n'est pas un sous-groupe invariant*, ainsi que le montre un calcul trivial. Si un système dynamique est *conservatif* dans un repère d'inertie, il en résulte qu'il peut *ne plus être conservatif dans un autre*. La formulation (17.24) du principe de Gibbs doit donc être élargie, pour devenir compatible avec la relativité galiléenne.

Nous proposons donc le principe suivant :

(17.7) Si un système dynamique est invariant par un sous-groupe de Lie G' du groupe de Galilée, les équilibres naturels du système constituent l'ensemble de Gibbs du groupe dynamique G'.

Soit $\mathcal{G}'$ l'algèbre de Lie G' ; on sait que $\mathcal{G}'$ est une sous-algèbre de Lie de celle de G, notée $\mathcal{G}$; un équilibre du système sera caractérisé par un élément Z de $\mathcal{G}'$, donc de $\mathcal{G}$; on pourra écrire

$$Z = \begin{bmatrix} j(\omega) & \beta & \gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{bmatrix} \qquad (17.78)$$

en utilisant les notations (13.4) ; Z parcourt l'ensemble Ω défini en (16.219) ; à chaque valeur de $\mathcal{Z}$ est associé un élément M du dual $\mathcal{G}'^*$ de $\mathcal{G}'$, valeur moyenne du moment μ ; on peut appliquer les formules (16.219), (16.220), qui généralisent les relations thermodynamiques (17.26), (17.27), (17.28). On voit que c'est Z (17.78) qui généralise la « température » ; le théorème d'isothermie (17.32) s'étend immédiatement : l'équilibre d'un système composé de plusieurs parties sans interactions s'obtient en attribuant à chaque composante un équilibre *correspondant à la même valeur de Z* ; l'entropie s, le potentiel de Planck z et le moment moyen M sont *additifs*. W

(17.79)

J.M. Souriau, Structure des systèmes dynamiques, Chapitre IV « Mécanique Statistique »

Trompette de Souriau

Lorsque le fait qu'on rencontre est en opposition avec une théorie régnante, il faut accepter le fait et abandonner la théorie, alors même que celle-ci, soutenue par de grands noms, est généralement adoptée

- Claude Bernard "Introduction à l'Étude de la Médecine Expérimentale"

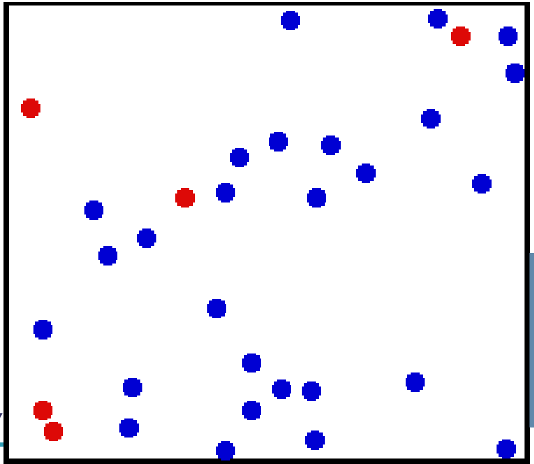
THALES
Building a future we can all trust

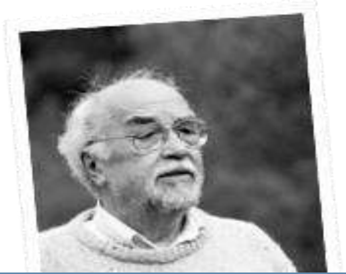
Symplectic Model of Souriau Lie Groups Thermodynamics

- Souriau Geometric (Planck) Temperature is an element of Lie Algebra of Dynamical Group (Galileo/Poincaré groups) acting on the system. Heat is an element of the dual of Lie algebra.

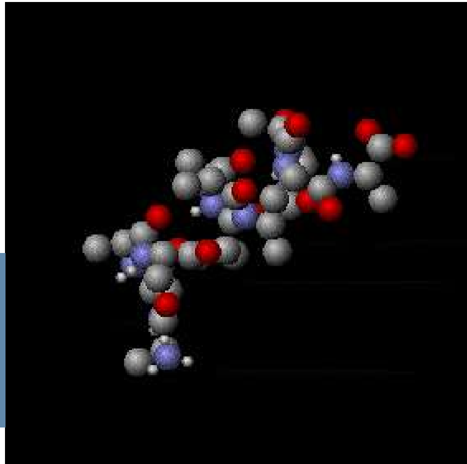
- Generalized Entropy is Legendre Transform of minus logarithm of Laplace Transform and is Casimir Function on Symplectic leaves (obtained by coadjoint orbit via moment map)

- Fisher(-Souriau) Metric is a Geometric Calorific Capacity (hessian of Massieu Potential)





Souriau formalism is fully covariant, with no special coordinates (covariance of Gibbs density wrt Dynamical Groups)



Jean-Marie Souriau Seminal Paper - 1974

Statistical Mechanics, Lie Group and Cosmology - 1st part: Symplectic Model of Statistical Mechanics

Jean-Marie Souriau

Abstract: The classical notion of Gibbs' canonical ensemble is extended to the case of a symplectic manifold on which a Lie group has a symplectic action ("dynamic group"). The rigorous definition given here makes it possible to extend a certain number of classical thermodynamic properties (temperature is here an element of the Lie algebra of the group, heat an element of its dual), notably inequalities of convexity. In the case of non-commutative groups, particular properties appear: the symmetry is spontaneously broken, certain relations of cohomological type are verified in the Lie algebra of the group. Various applications are considered (rotating bodies, covariant or relativistic statistical Mechanics). [These results specify and complement a study published in an earlier work (*), which will be designated by the initials SSD].

(*) Souriau, J.-M., Structure des systèmes dynamique. Dunod, collection Dunod Université, Paris 1969. http://www.jmsouriau.com/structure_des_systemes_dynamiques.htm

Souriau, J-M., Mécanique statistique, groupes de Lie et cosmologie, Colloques Internationaux C.N.R.S., n°237 – Géométrie symplectique et physique mathématique, pp.59-113, 1974

English translation by F. Barbaresco: https://www.academia.edu/42630654/Statistical_Mechanics_Lie_Group_and_Cosmology_1_st_part_Symplectic_Model_of_Statistical_Mechanics

Charles-Michel Marle (Corps des Mines, Sorbonne Université)

On Gibbs states of mechanical systems with symmetries

Charles-Michel Marle

Gibbs states for the Hamiltonian action of a Lie group on a symplectic manifold were studied, and their possible applications in Physics and Cosmology were considered, by the French mathematician and physicist Jean-Marie Souriau. They are presented here with detailed proofs of all the stated results. Using an adaptation of the cross product for pseudo-Euclidean three-dimensional vector spaces, we present several examples of such Gibbs states, together with the associated thermodynamic functions, for various two-dimensional symplectic manifolds, including the pseudo-spheres, the Poincaré disk and the Poincaré half-plane.

https://arxiv.org/abs/2012.00582

X 53

Symplectic Geometry & Foliations Theory Epic

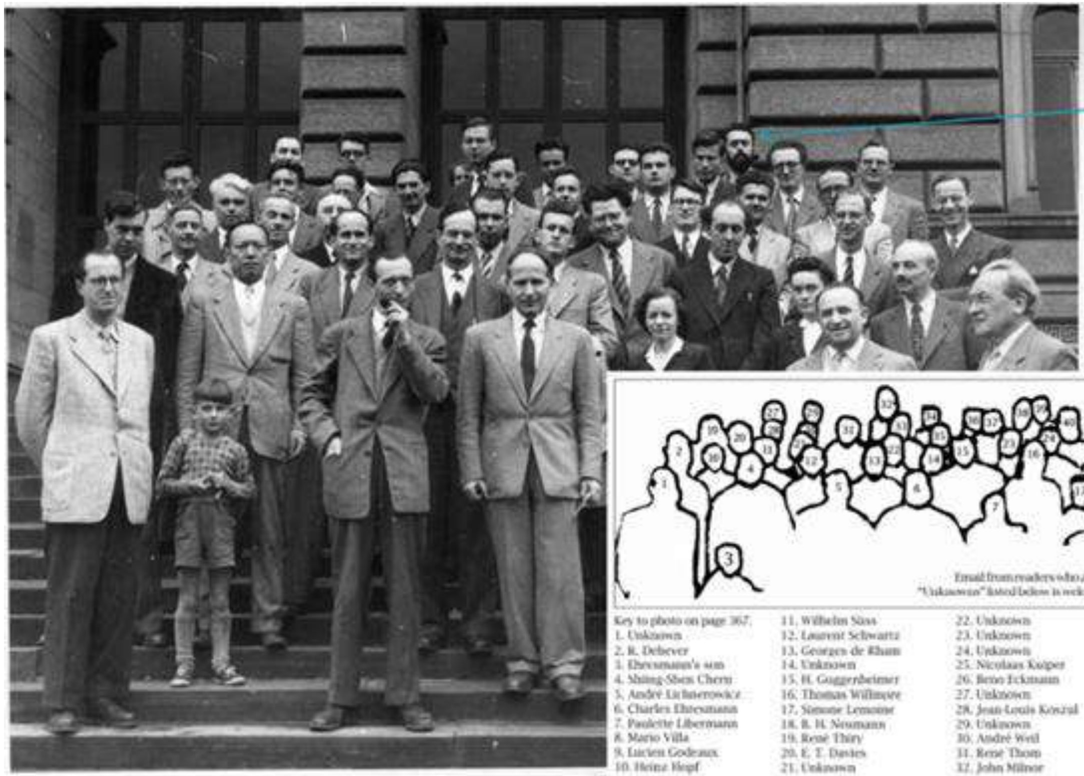
F. Barbaresco, *Esthétique structurelle des feuilletages symplectiques du mouvement, de la chaleur et de l'information : la quête romantique de Jean-Marie Souriau de Carthage à Massilia ou le triptyque de la Nature des choses (De Rerum Natura) « Esthétisme – Structure – Mouvement »*, **ENS/IRCAM MAMUPHI seminar, to be published by Spartacus Editor for MAMUPHI anniversary https://spartacus-idh.com/**

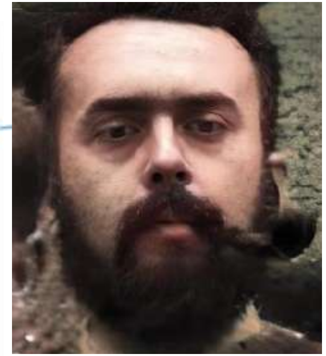
MAMUPHI video: https://www.youtube.com/watch?v=dWyWXubGfXA

MAMUPHI Slides: http://www.entretemps.asso.fr/2022-2023/Barbaresco.pdf

https://www.academia.edu/112471996/Structural_aesthetics_of_the_symplectic_foliations_of_movement_heat_and_information_the_romantic_quest_of_Jean_Marie_Souriau_from_Carthage_to_Massilia_or_the_triptych_of_the_Nature_of_things_De_Rerum_Natura_Aesthetics_Structure_Movement_

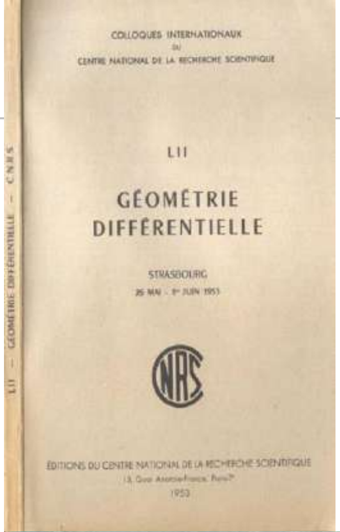
Strasbourg 1953





Email from readers who are able to identify any of the "Unknowns" listed below is welcome at [illegible]

Key to photo on page 367.
1. Unknown
2. R. Debever
3. Ehresmann's son
4. Shiing-Shen Chern
5. André Lichnerowicz
6. Charles Ehresmann
7. Paulette Libermann
8. Mario Villa
9. Lucien Godeaux
10. Heinz Hopf
11. Wilhelm Süss
12. Laurent Schwartz
13. Georges de Rham
14. Unknown
15. H. Guggenheimer
16. Thomas Willmore
17. Simone Lemoine
18. B. H. Neumann
19. René Thiry
20. E. T. Davies
21. Unknown
22. Unknown
23. Unknown
24. Unknown
25. Nicolaas Kuiper
26. Beno Eckmann
27. Unknown
28. Jean-Louis Koszul
29. Unknown
30. André Weil
31. René Thom
32. John Milnor
33. Marcel Berger
34. Unknown
35. Bernard Malgrange
36. Daniel Bernard
37. André Aragnol
38. G. Legrand
39. Jean-Marie Souriau
40. Unknown
41. Georges Reeb
42. Unknown



COLLOQUES INTERNATIONAUX
DU
CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

LII

GÉOMÉTRIE
DIFFÉRENTIELLE

STRASBOURG
26 MAI - 1er JUIN 1953

ÉDITIONS DU CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE
13, Quai Anatole-France, Paris 7e
1953

LII — GÉOMÉTRIE DIFFÉRENTIELLE — C.N.R.S.

PUBLICATIONS SCIENTIFIQUES
DE L'UNIVERSITÉ D'ALGER

SÉRIE A SCIENCES
MATHÉMATIQUES

DÉCEMBRE 1954. — Volume I, N° 2.

Pages

L. GODEAUX. — Structure des points de diramation des surfaces multiples 223
J.-M. SOURIAU. — Equations canoniques et géométrie symplectique 239
R. de POSSEL et J. VALENSI. — Sur le sillage d'une plaque perméable 267
M. LAZARD. — Sur les algèbres enveloppantes universelles de certaines algèbres de Lie 281
R. MEYNIEUX. — Configuration de Milne, Dixon et Morton en géométrie de caractéristique 3 ou 2 295
P. HONNORAT. — Sur une approximation dans le problème du bipendule 303
B. d'ORGEVAL. — A propos des surfaces du 4e ordre possédant des points doubles inflexionnels 307
B. d'ORGEVAL. — Surfaces elliptiques avec un faisceau de courbes de genre 4 313

IMPRIMERIE DURAND CHARTRES — 9, RUE FULBERT

Lagrange 2-form rediscovered by Jean-Marie Souriau

- Rewriting equations of classical mechanics in phase space $\begin{pmatrix} r \\ v \end{pmatrix}$

$$m\frac{d^2r}{dt^2}=F \quad \Longrightarrow \quad m\frac{dv}{dt}=F \quad \text{et} \quad v=\frac{dr}{dt}$$

- Souriau rediscovered that Lagrange had considered the evolution space: $y=\begin{pmatrix} t \\ r \\ v \end{pmatrix}\in V$

$$\begin{cases} m\delta v - F\delta t = 0 \\ \delta r - v\delta t = 0 \end{cases}$$

- **A dynamic system is represented by a foliation**. This foliation is determined by an antisymmetric covariant 2nd order tensor σ, called the Lagrange (-Souriau) form, a bilinear operator on the tangent vectors of V.

$$\sigma(\delta y)(\delta' y)=\langle m\delta v - F\delta t, \delta' r - v\delta' t\rangle - \langle m\delta' v - F\delta' t, \delta r - v\delta t\rangle \qquad \delta y=\begin{pmatrix} \delta t \\ \delta r \\ \delta v \end{pmatrix} \quad \text{et} \quad \delta' y=\begin{pmatrix} \delta' t \\ \delta' r \\ \delta' v \end{pmatrix}$$

- *In the Lagrange-Souriau model,* σ *is a 2-form on the evolution space V, and the differential equation of motion implies:* $\delta y \in \varepsilon$

$$\sigma(\delta y)(\delta' y)=0 \ , \ \forall \delta' y \qquad \sigma(\delta y)=0 \ \text{ ou } \ \delta y \in \ker(\sigma)$$

Souriau Work Roots: François Gallissot Theorem

- **Gallissot Theorem**: There are 3 types of differential forms generating the equations of a material point motion, **invariant by the action of the Galileo group**

$$A:\begin{cases} s=\dfrac{1}{2m}\displaystyle\sum_{i=1}^{3}\left(mdv_i-F_idt\right)^2 \\ e=\dfrac{m}{2}\displaystyle\sum_{j=1}^{3}\left(dx_j-v_jdt\right)^2 \end{cases}$$

F. Gallissot, Les formes extérieures en Mécanique (*Thèse*), Durand, Chartres, 1954.

$$B: f=\sum_{1}^{3}\delta_{ij}\left(dx_i-v_idt\right)\left(mdv_j-F_jdt\right) \text{ with } \delta_{ij} \text{ krönecker symbol}$$

$$C:\ \omega=\sum_{1}^{3}\delta_{ij}\left(mdv_i-F_idt\right)\wedge\left(dx_j-v_jdt\right)$$

- $d\omega=0$ constrained the Pfaff form $\delta_{ij}F_idx_j$ to be closed and to be reduced to the differential of U : $C\Rightarrow\omega=m\delta_{ij}dv_i\wedge dx_j-dH\wedge dt$ with $H=T-U$ and $T=1/2\sum_{i=1}^{3}m\left(v_i\right)^2$
- It proves that ω has an exterior differential $d\omega$ generating **Poincaré-Cartan Integral invariant:**

$$d\omega=\sum_{i=1}^{3}mv_idx_j-Hdt$$

Interior/Exterior Products and Lie derivative

- $i_V\omega$ is the (p-1)-form on X obtained by inserting $V(x)$ as the first argument of ω :

Interior product : $i_V\omega\left(v_2,\cdots v_p\right)=\omega\left(V(x),v_2,\cdots,v_p\right)$

- $\theta\wedge\omega$ is the (p + 1)-form on X where ω is a p-form and θ is a 1-form on X :

Exterior product : $\theta\wedge\omega\left(v_0,\cdots,v_p\right)=\sum_{i=0}^{p}(-1)^i\,\theta(v_i)\omega\left(v_0,\cdots,\hat{v}_i,\cdots,v_p\right)$

(where the hat indicates a term to be omitted).

- $L_V\omega$ is a p-form on X ,and $L_V\omega=0$ if the flow of V consists of symmetries of ω:

Lie derivative : $L_V\omega\left(v_1,\cdots,v_p\right)=\left.\frac{d}{dt}e^{tV*}\omega\left(v_1,\cdots,v_p\right)\right|_{t=0}$

Exterior derivative and E.Cartan, H. Cartan & S. Lie formulas

- $d\omega$ is the (p+1)-form on X defined by taking the ordinary derivative of ω and then antisymmetrizing:

Exterior derivative : $$d\omega\left(v_0,\cdots,v_p\right)=\sum_{i=0}^{p}(-1)^i\frac{\partial\omega}{\partial x}(v_i)(v_0,\cdots,\hat{v}_i,\cdots,v_p)$$

$$p=0,[d\omega]_i=\partial_i\omega \;\;;\;\; p=1,[d\omega]_{ij}=\partial_i\omega_j-\partial_j\omega_i \;\;;\;\; p=2,[d\omega]_{ijk}=\partial_i\omega_{jk}+\partial_j\omega_{ki}+\partial_k\omega_{ij}$$

- The properties of the exterior and Lie Derivative are the following:

$$L_V\omega=di_V\omega+i_Vd\omega$$ (E. Cartan)

$$i_{[U,V]}\omega=i_VL_U\omega-L_Ui_V\omega$$ (H.Cartan)

$$L_{[U,V]}\omega=L_VL_U\omega-L_UL_V\omega$$ (S. Lie)

Souriau Moment Map (1/2)

- Let $\left(X,\sigma\right)$ be a connected symplectic manifold.
- A vector field η on X is called symplectic if its flow preserves the 2-form :

$$L_\eta \sigma = 0$$

- If we use Elie Cartan's formula, we can deduce that :

$$L_\eta \sigma = d i_\eta \sigma + i_\eta d\sigma = 0$$

- but as $d\sigma = 0$ then $d i_\eta \sigma = 0$. We observe that the 1-form $i_\eta \sigma$ is closed.
- When this 1-form is exact, there is a smooth function $x \mapsto H$ on X with:

$$i_\eta \sigma = -dH$$

- This vector field η is called Hamiltonian and could be define as s symplectic gradient :

$$\eta = \nabla_{Symp} H$$

Souriau Moment Map (2/2)

$$di_\eta \sigma = 0 \qquad \boxed{i_\eta \sigma = -dH}$$

- We define the Poisson bracket of two functions H , H' by :

$$\{H, H'\} = \sigma(\eta, \eta') = \sigma\left(\nabla_{Symp} H', \nabla_{Symp} H\right)$$

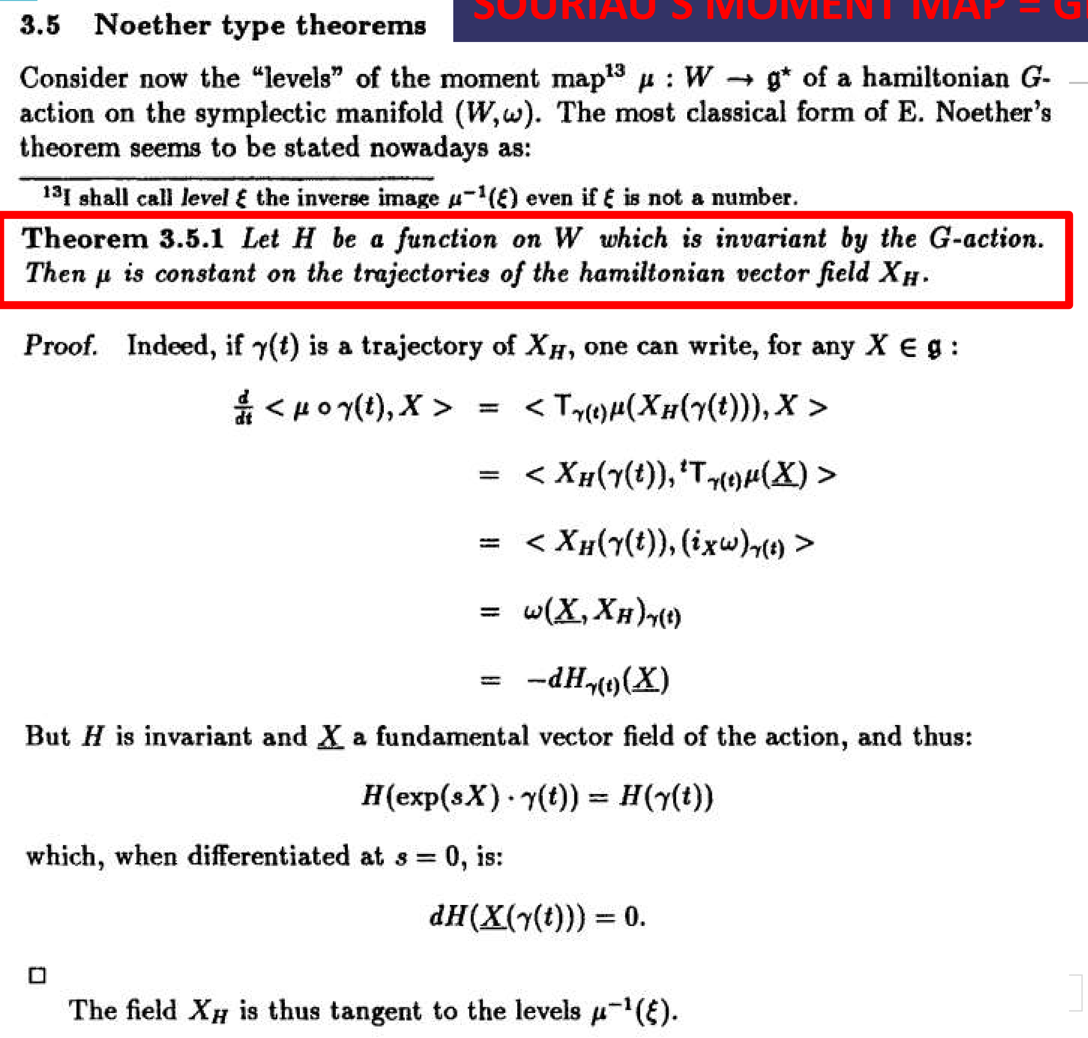
with $i_\eta \sigma = -dH$ and $i_{\eta'} \sigma = -dH'$

- Let a Lie group G that acts on X and that also preserve σ .

- A moment map exists if these infinitesimal generators are actually hamiltonian, so that a map exists:

$\Phi : X \to \mathfrak{g}^*$ with $i_{Z_X} \sigma = -dH_Z$ where $H_Z = \left\langle \Phi(x), Z \right\rangle$

SOURIAU'S MOMENT MAP = GEOMETRIZATION OF NOETHER THEOREM



3.5 Noether type theorems

Consider now the "levels" of the moment map[13] $\mu : W \to \mathfrak{g}^\star$ of a hamiltonian G-action on the symplectic manifold (W,ω). The most classical form of E. Noether's theorem seems to be stated nowadays as:

---

[13] I shall call *level* ξ the inverse image $\mu^{-1}(\xi)$ even if ξ is not a number.

Theorem 3.5.1 *Let H be a function on W which is invariant by the G-action. Then μ is constant on the trajectories of the hamiltonian vector field X_H.*

Proof. Indeed, if $\gamma(t)$ is a trajectory of X_H, one can write, for any $X \in \mathfrak{g}$:

$$\begin{aligned}
\tfrac{d}{dt} < \mu \circ \gamma(t), X > &= < T_{\gamma(t)}\mu(X_H(\gamma(t))), X > \\
&= < X_H(\gamma(t)), {}^{t}T_{\gamma(t)}\mu(\underline{X}) > \\
&= < X_H(\gamma(t)), (i_X\omega)_{\gamma(t)} > \\
&= \omega(\underline{X}, X_H)_{\gamma(t)} \\
&= -dH_{\gamma(t)}(\underline{X})
\end{aligned}$$

But H is invariant and $\underline{X}$ a fundamental vector field of the action, and thus:

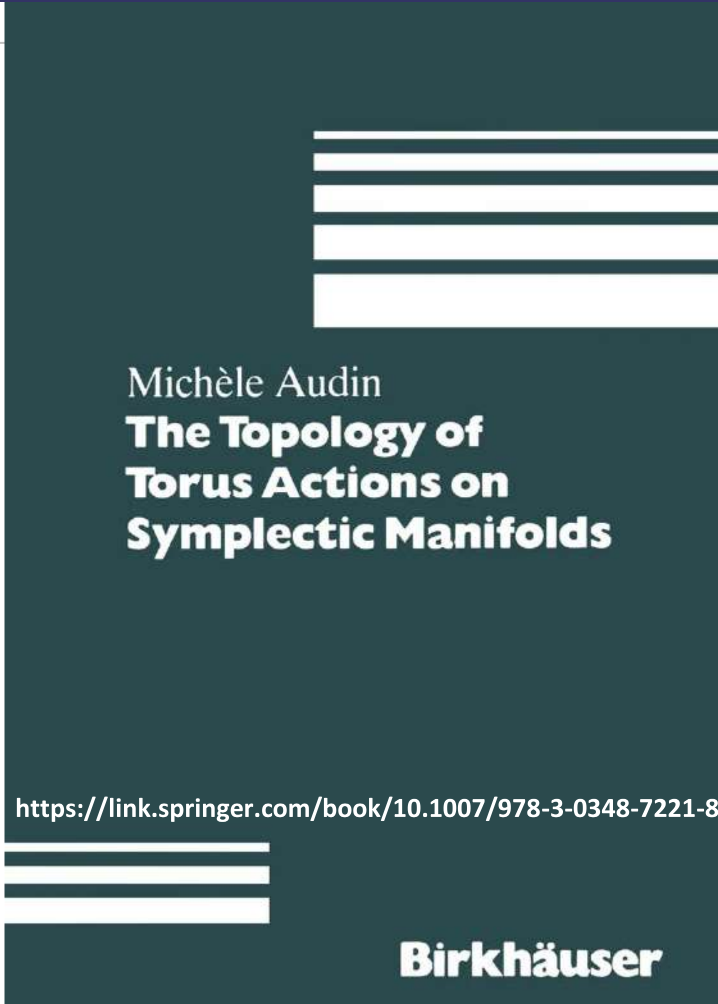
$$H(\exp(sX)\cdot\gamma(t)) = H(\gamma(t))$$

which, when differentiated at $s = 0$, is:

$$dH(\underline{X}(\gamma(t))) = 0.$$

☐

The field X_H is thus tangent to the levels $\mu^{-1}(\xi)$.



Michèle Audin

The Topology of Torus Actions on Symplectic Manifolds

https://link.springer.com/book/10.1007/978-3-0348-7221-8

Birkhäuser

Coadjoint operator and Coadjoint Orbits (Kirillov Representation)

Lie Group Adjoint Representation

- the adjoint representation of a Lie group Ad_g is a way of representing its elements as linear transformations of the Lie algebra, considered as a vector space

$$Ad_g = \left(d\Psi_g\right)_e : \mathfrak{g} \to \mathfrak{g}$$

$$X \mapsto Ad_g(X) = gXg^{-1}$$

$$\mathrm{Ad}_g(X) = \frac{\mathrm{d}}{\mathrm{d}t} g e^{tX} g^{-1}\Big|_{t=0} \in T_e G$$

$$\Psi : G \to Aut\left(G\right)$$

$$g \mapsto \Psi_g\left(h\right) = ghg^{-1}$$

$$ad = T_e Ad : T_e G \to End\left(T_e G\right)$$

$$X, Y \in T_e G \mapsto ad_X(Y) = \left[X, Y\right]$$

Lie Group Co-Adjoint Representation

- the coadjoint representation of a Lie group Ad_g^*, is the dual of the adjoint representation ($\mathfrak{g}^*$ denotes the dual space to $\mathfrak{g}$):

$$\forall g \in G, Y \in \mathfrak{g}, F \in \mathfrak{g}^*, \text{ then } \left\langle Ad_g^* F, Y \right\rangle = \left\langle F, Ad_{g^{-1}} Y \right\rangle$$

$$K = Ad_g^* = \left(Ad_{g^{-1}}\right)^* \quad \text{and} \quad K_*(X) = -\left(ad_X\right)^*$$

Coadjoint operator and Coadjoint Orbits (Kirillov Representation)

Co-adjoint Orbits as Homogeneous Symplectic Manifold by KKS 2-form

- A coadjoint orbit: $O_F = \left\{ Ad_g^* F, g \in G \right\}$ subset of $\mathfrak{g}^*, F \in \mathfrak{g}^*$

  carry a natural homogeneous symplectic structure by a closed G-invariant 2-form:

$$\sigma_\Omega \left(K_{*X} F, K_{*Y} F \right) = B_F \left(X, Y \right) = \left\langle F, \left[X, Y \right] \right\rangle, X, Y \in \mathfrak{g}$$

- The coadjoint action on O_F is a Hamiltonian G-action with moment map $\Omega \rightarrow \mathfrak{g}^*$

Souriau Foundamental Theorem « *Every symplectic manifold is a coadjoint orbit* » is based on classification of symplectic homogeneous Lie group actions by Souriau, Kostant and Kirillov

$$g \in G$$

Lie Group

$$O_F = \left\{ Ad_g^* F, g \in G, F \in \mathfrak{g}^* \right\}$$

Coadjoint Orbit

(action of Lie Group on dual Lie algebra)

$$\sigma_\Omega \left(ad_F X, ad_F Y \right) = \left\langle F, \left[X, Y \right] \right\rangle$$

$$X, Y \in \mathfrak{g}, F \in \mathfrak{g}^*$$

Homogeneous Symplectic Manifold

(a smooth manifold with a closed differential 2-form σ, such that dσ=0, where the Lie Group acts transitively)

To make statistics on Lie Groups, migrate on the symplectic manifold generated by coadjoint orbits to capture symmetries

Bedrock of Information Geometry

Jean-Marie Souriau (ENS 1942)

Jean-Louis Koszul (ENS 1940)

Information Geometry & Machine Learning : Legendre structure

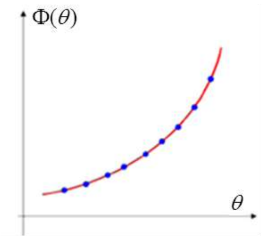
Legendre Transform, Dual Potentials & Fisher Metric

- **S.I. Amari** has proved that the Riemannian metric in an exponential family is the **Fisher information matrix** defined by:

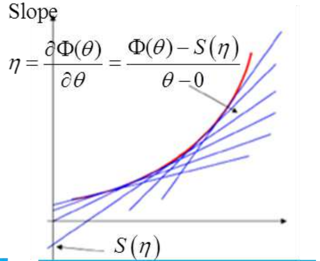
$$g_{ij} = -\left[\frac{\partial^2 \Phi}{\partial \theta_i \partial \theta_j}\right]_{ij} \quad \text{with} \quad \Phi(\theta) = -\log \int_R e^{-\langle \theta, y \rangle} d\lambda$$

- and the dual potential, the **Shannon entropy**, is given by the **Legendre transform**:

$$S(\eta) = \langle \theta, \eta \rangle - \Phi(\theta) \quad \text{with} \quad \eta_i = \frac{\partial \Phi(\theta)}{\partial \theta_i} \quad \text{and} \quad \theta_i = \frac{\partial S(\eta)}{\partial \eta_i}$$



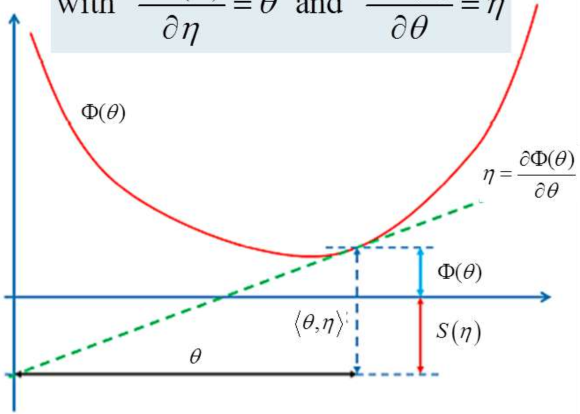
Classical Geometry
(curve is given by a continuum of points)



Plücker Geometry
(curve is given by the envelop of its tangents)

$$S(\eta) = \langle \theta, \eta \rangle - \Phi(\theta)$$

$$\text{with} \quad \frac{\partial S(\eta)}{\partial \eta} = \theta \quad \text{and} \quad \frac{\partial \Phi(\theta)}{\partial \theta} = \eta$$



Fisher Metric and Koszul 2 form on sharp convex cones

Koszul-Vinberg Characteristic Function, Koszul Forms

- **J.L. Koszul** and **E. Vinberg** have introduced an affinely invariant Hessian metric on a sharp convex cone through its **characteristic function**

$$\Phi_\Omega(\theta) = -\log \int_{\Omega^*} e^{-\langle \theta, y\rangle} dy = -\log \psi_\Omega(\theta) \quad \text{with } \theta \in \Omega \text{ sharp convex cone}$$

$$\psi_\Omega(\theta) = \int_{\Omega^*} e^{-\langle \theta, y\rangle} dy \quad \text{with Koszul-Vinberg Characteristic function}$$

Jean-Louis Koszul

- **1st Koszul form α**: $\alpha = d\Phi_\Omega(\theta) = -d\log \psi_\Omega(\theta)$
- **2nd Koszul form γ**: $\gamma = D\alpha = Dd\log \psi_\Omega(\theta)$

$$\left(Dd\log \psi_\Omega(x)\right)(u) = \frac{1}{\psi_\Omega(u)^2}\left[\int_{\Omega^*} F(\xi)^2 d\xi . \int_{\Omega^*} G(\xi)^2 d\xi - \left(\int_{\Omega^*} F(\xi).G(\xi) d\xi\right)^2\right] > 0 \quad \text{with} \quad F(\xi) = e^{-\frac{1}{2}\langle x,\xi\rangle} \quad \text{and} \quad G(\xi) = e^{-\frac{1}{2}\langle x,\xi\rangle}\langle u,\xi\rangle$$

- Diffeomorphism: $\eta = \alpha = -d\log \psi_\Omega(\theta) = \int_{\Omega^*} \xi p_\theta(\xi) d\xi \quad \text{with } p_\theta(\xi) = \dfrac{e^{-\langle \xi,\theta\rangle}}{\int_{\Omega^*} e^{-\langle \xi,\theta\rangle} d\xi}$
- Legendre transform: $S_\Omega(\eta) = \langle \theta, \eta\rangle - \Phi_\Omega(\theta) \quad \text{with} \quad \eta = d\Phi_\Omega(\theta) \quad \text{and} \quad \theta = dS_\Omega(\eta)$

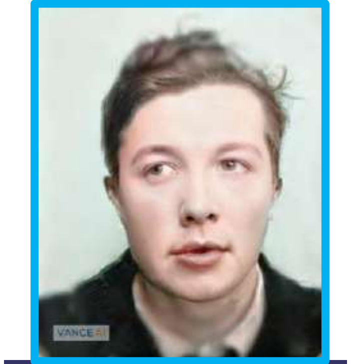
Fisher Metric and Souriau 2-form: Lie Groups Thermodyamics

Statistical Mechanics, Dual Potentials & Fisher Metric

- In geometric statistical mechanics, **J.M. Souriau** has developed a "**Lie groups thermodynamics**" of dynamical systems where the (maximum entropy) **Gibbs density is covariant** with respect to the action of the Lie group. In the Souriau model, previous structures of information geometry are preserved:

$$I(\beta) = -\frac{\partial^2 \Phi}{\partial \beta^2} \text{ with } \Phi(\beta) = -\log \int_M e^{-\langle \beta, U(\xi) \rangle} d\lambda \qquad U : M \to \mathfrak{g}^*$$

$$S(Q) = \langle \beta, Q \rangle - \Phi(\beta) \text{ with } Q = \frac{\partial \Phi(\beta)}{\partial \beta} \in \mathfrak{g}^* \text{ and } \beta = \frac{\partial S(Q)}{\partial Q} \in \mathfrak{g}$$



Jean-Marie Souriau

- In the Souriau **Lie groups thermodynamics** model, β is a "geometric" (Planck) temperature, element of Lie algebra $\mathfrak{g}$ of the group, and Q is a "geometric" heat, element of dual Lie algebra $\mathfrak{g}^*$ of the group.

Fisher-Souriau Metric as a non-null Cohomology extension of KKS 2 form (Kirillov-Kostant-Souriau 2 form)

- Souriau definition of Fisher Metric is related to the extension of KKS 2-form (Kostant-Kirillov-Souriau) in case of non-null Cohomogy:

Souriau-Fisher Metric

$$I(\beta)=\left[g_\beta\right] \text{ with } g_\beta\left([\beta,Z_1],[\beta,Z_2]\right)=\tilde{\Theta}_\beta\left(Z_1,[\beta,Z_2]\right)$$

$$\text{with } \tilde{\Theta}_\beta\left(Z_1,Z_2\right)=\tilde{\Theta}\left(Z_1,Z_2\right)+\left\langle Q,[Z_1,Z_2]\right\rangle$$

Non-null cohomology: aditional term from Souriau Cocycle

Equivariant KKS 2 form

$$\tilde{\Theta}(X,Y)=J_{[X,Y]}-\left\{J_X,J_Y\right\} \text{ with } J(x):M\rightarrow\mathfrak{g}^* \text{ such that } J_X(x)=\left\langle J(x),X\right\rangle, X\in\mathfrak{g}$$

$$\tilde{\Theta}\left(X,Y\right):\mathfrak{g}\times\mathfrak{g}\rightarrow\Re \qquad \text{with } \Theta(X)=T_e\theta\left(X(e)\right)$$

$$\text{X,Y}\mapsto\left\langle\Theta(X),Y\right\rangle$$

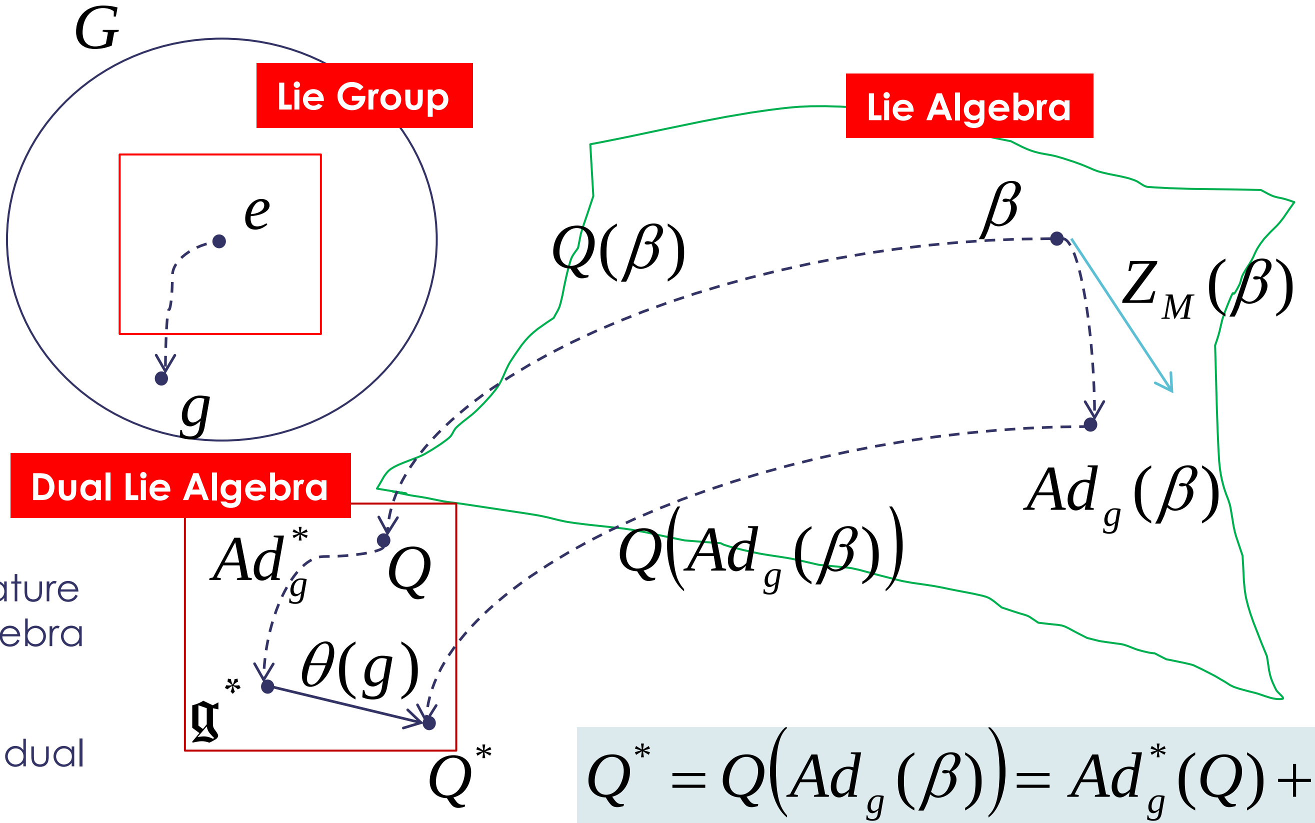
$$\tilde{\Theta}\left(\beta,Z\right)+\left\langle Q,[\beta,Z]\right\rangle=0$$

$$\beta\in Ker\,\tilde{\Theta}_\beta$$

Souriau Fundamental
Equation of Lie Group Thermodynamics

$$Q\left(Ad_g(\beta)\right)=Ad_g^*(Q)+\theta\left(g\right)$$

Fundamental Souriau Theorem

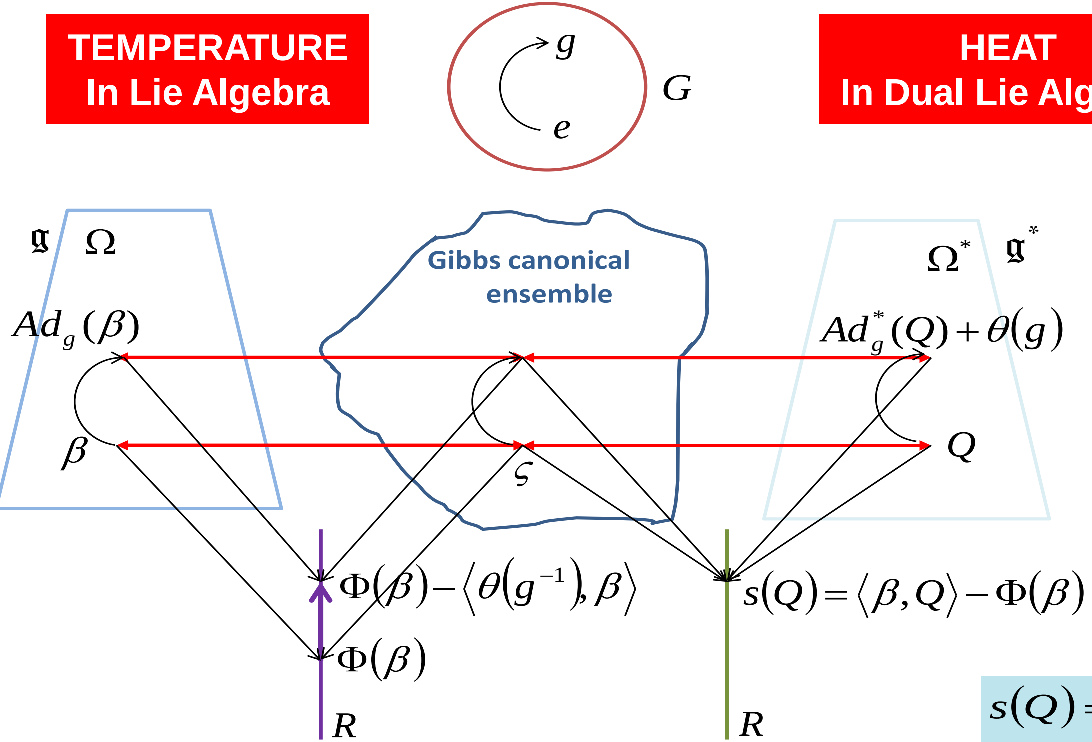


β : (Planck) température element of Lie algebra

Q : Heat, element of dual Lie Algebra

$$Q^* = Q(Ad_g(\beta)) = Ad_g^*(Q) + \theta(g)$$

Souriau-Fisher Metric & Souriau Lie Groups Thermodynamics: Bedrock for Lie Group Machine Learning



TEMPERATURE
In Lie Algebra

HEAT
In Dual Lie Algebra

Logarithm of Partition Function
(Massieu Characteristic Function)

Entropy

Fisher Metric

$$g_\beta([\beta,Z_1],[\beta,Z_2])=\tilde{\Theta}_\beta(Z_1,[\beta,Z_2])\geq 0$$

$$I(\beta)=I\left(Ad_g(\beta)\right)=-\frac{\partial^2\Phi}{\partial\beta^2}=\frac{\partial^2\log\int\limits_M e^{-\langle\beta,U(\xi)\rangle}d\lambda}{\partial\beta^2}$$

Entropy invariant under the action of the group

Legendre **Clairaut**

$$s(Q)=\langle\beta,Q\rangle-\Phi(\beta)=\langle\Theta^{-1}(Q),Q\rangle-\Phi\left(\Theta^{-1}(Q)\right)$$

$$Q=\Theta(\beta)=\frac{\partial\Phi}{\partial\beta}\in\mathfrak{g}^*$$

$$\beta=\Theta^{-1}(Q)\in\mathfrak{g}$$

Gallileo Group & Algebra & V. Bargman Central extensions

- Symplectic cocycles of the Galilean group: V. Bargmann (Ann. Math. 59, 1954, pp 1–46) has proven that the symplectic cohomology space of the Galilean group is one-dimensional.

- Gallileo Lie Group & Algebra

$$\begin{cases} \vec{x}' = R.\vec{x} + \vec{u}.t + \vec{w} \\ t' = t + e \end{cases}$$

$$\vec{x}, \vec{u} \;\text{ and }\; \vec{w} \in R^3, e \in R^+$$

$$R \in SO(3)$$

$$\begin{bmatrix} \vec{x}' \\ t' \\ 1 \end{bmatrix} = \begin{bmatrix} R & \vec{u} & \vec{w} \\ 0 & 1 & e \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{x} \\ t \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \vec{\omega} & \vec{\eta} & \vec{\gamma} \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{bmatrix}, \begin{cases} \vec{\eta} \;\text{ and }\; \vec{\gamma} \in R^3, \varepsilon \in R^+ \\ \vec{\omega} \in so(3) : \vec{x} \mapsto \vec{\omega} \times \vec{x} \end{cases}$$

- Bargmann Central extension:

$$\begin{bmatrix} R & \vec{u} & 0 & \vec{w} \\ 0 & 1 & 0 & e \\ -\vec{u}^t R & -\dfrac{\|\vec{u}\|^2}{2} & 1 & f \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Souriau Gibbs states for Hamiltonian actions of subgroups of the Galilean group

Souriau Gibbs states for one-parameter subgroups of the Galilean group

- **Souriau Result:** Action of the full Galilean group on the space of motions of an isolated mechanical system is not related to any Equilibrium Gibbs state (the open subset of the Lie algebra, associated to this Gibbs state, is empty)
- The **1-parameter subgroup of the Galilean group** generated by β element of Lie Algebra, is the set of matrices

$$\exp(\tau\beta)=\begin{pmatrix} A(\tau) & \vec{b}(\tau) & \vec{d}(\tau) \\ 0 & 1 & \tau\varepsilon \\ 0 & 0 & 1 \end{pmatrix} \text{ with } \begin{cases} A(\tau)=\exp\left(\tau j(\vec{\omega})\right) \text{ and } \vec{b}(\tau)=\left(\sum\limits_{i=1}^{\infty}\dfrac{\tau^i}{i!}\left(j(\vec{\omega})\right)^{i-1}\right)\vec{\alpha} \\ \vec{d}(\tau)=\left(\sum\limits_{i=1}^{\infty}\dfrac{\tau^i}{i!}\left(j(\vec{\omega})\right)^{i-1}\right)\vec{\delta}+\varepsilon\left(\sum\limits_{i=2}^{\infty}\dfrac{\tau^i}{i!}\left(j(\vec{\omega})\right)^{i-2}\right)\vec{\alpha} \end{cases}$$

$$\beta=\begin{pmatrix} j(\vec{\omega}) & \vec{\alpha} & \vec{\delta} \\ 0 & 1 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix}\in\mathfrak{g}$$

Souriau Thermodynamics of butter churn (device used to convert cream into butter) or “La Thermodynamique de la crémière”

If we consider the case of the centrifuge

$$\vec{\omega} = \omega \vec{e}_z \,, \vec{\alpha} = 0 \ \text{ and } \ \vec{\delta} = 0$$

$$Rotation\ speed : \frac{\omega}{\varepsilon}$$

$$f_i\left(\vec{r}_{i0}\right) = -\frac{\omega^2}{2\varepsilon^2}\left\|\vec{e}_z \times \vec{r}_{i0}\right\|^2$$

$$\text{with } \ \Delta = \left\|\vec{e}_z \times \vec{r}_{i0}\right\| \ \text{ distance to axis z}$$

> **“The angular momentum is imparted to the gas when the molecules collide with the rotating walls, which changes the Maxwell distribution at every point, shifting its origin. The walls play the role of an angular momentum reservoir. Their motion is characterized by a certain angular velocity, and the angular velocities of the fluid and of the walls become equal at equilibrium, exactly like the equalization of the temperature through energy exchanges”.** – Roger Balian

$$\rho_i(\beta) = \frac{1}{P_i(\beta)}\exp\left(-\langle J_i, \beta\rangle\right) = cst.\exp\left(-\frac{1}{2m_i\kappa T}\left\|\vec{p}_{i0}\right\|^2 + \frac{m_i}{2\kappa T}\left(\frac{\omega}{\varepsilon}\right)^2\Delta^2\right)$$

- the behaviour of a gas made of point particles of various masses in a centrifuge rotating at a constant angular velocity (the heavier particles concentrate farther from the rotation axis than the lighter ones)

$$\frac{\omega}{\varepsilon}$$

Link with Classical Thermodynamics

We have the reciprocal formula:

$Q = \dfrac{\partial \Phi}{\partial \beta}$	$\beta = \dfrac{\partial S}{\partial Q}$
$S(Q) = \left\langle \dfrac{\partial \Phi}{\partial \beta}, \beta \right\rangle - \Phi$	$\Phi(\beta) = \left\langle Q, \dfrac{\partial S}{\partial Q} \right\rangle - S$

For Classical Thermodynamics (Time translation only), we recover the definition of Boltzmann-Clausius Entropy:

$$\begin{cases} \beta = \dfrac{\partial S}{\partial Q} \\ \beta = \dfrac{1}{T} \end{cases} \Rightarrow dS = \frac{dQ}{T}$$

Souriau Model of Covariant Gibbs Density

Covariant Souriau-Gibbs density

- Souriau has then defined a Gibbs density that is covariant under the action of the group:

$$p_{Gibbs}(\xi) = e^{\Phi(\beta)-\langle U(\xi),\beta\rangle} = \frac{e^{-\langle U(\xi),\beta\rangle}}{\int\limits_M e^{-\langle U(\xi),\beta\rangle} d\lambda_\omega}$$

$$\text{with } \; \Phi(\beta) = -\log \int\limits_M e^{-\langle U(\xi),\beta\rangle} d\lambda_\omega$$

$$Q = \frac{\partial \Phi(\beta)}{\partial \beta} = \frac{\int\limits_M U(\xi) e^{-\langle U(\xi),\beta\rangle} d\lambda_\omega}{\int\limits_M e^{-\langle U(\xi),\beta\rangle} d\lambda_\omega} = \int\limits_M U(\xi) p(\xi) d\lambda_\omega$$

- We can express the Gibbs density with respect to Q by inverting the relation $Q = \frac{\partial \Phi(\beta)}{\partial \beta} = \Theta(\beta)$. Then $p_{Gibbs,Q}(\xi) = e^{\Phi(\beta)-\langle U(\xi),\Theta^{-1}(Q)\rangle}$ with $\beta = \Theta^{-1}(Q)$

ENTROPY as Casimir Function in Coadjoint Representation Constant on Symplectic Leaves & Density of Maximum Entropy on Lie Groups


Jean-Marie Souriau

Souriau Entropy Invariance

Casimir Invariant Function in coadjoint representation

- We observe that Souriau Entropy $S(Q)$ defined on coadjoint orbit of the group has a property of invariance :

$$S\left(Ad_g^{\#}(Q)\right)=S(Q)$$

- with respect to Souriau affine definition of coadjoint action:

$$Ad_g^{\#}(Q)=Ad_g^{*}(Q)+\theta(g)$$

- where $\theta(g)$ is called the Souriau cocyle.

$$Q\left(Ad_g(\beta)\right)=Ad_g^{*}(Q)+\theta(g)$$

$$S\left(Q\left(Ad_g(\beta)\right)\right)=S(Q)$$

New Entropy Definition:
Casimir Function in Coadjoint Representation Invariant under the action of the Group

ROTATION OF A RIGID BODY
IN QUANTUM MECHANICS

ACADEMISCH PROEFSCHRIFT

HENDRIK BRUGT GERHARD CASIMIR

Hendrik Casimir
(Thesis supervised by
Niels Bohr & Paul Ehrenfest)

H.B.G. Casimir, On the Rotation of a Rigid Body in Quantum Mechanics, Doctoral Thesis, Leiden, 1931.

Souriau Entropy and Casimir Invariant Function

Demo

- if we consider the heat expression $Q=\frac{\partial\Phi}{\partial\beta}$, that we can write $\delta\Phi-\langle Q,\delta\beta\rangle=0$.
- For each $\delta\beta$ tangent to the orbit, and so generated by an element Z of the Lie algebra, if we consider the relation $\Phi\left(Ad_g(\beta)\right)=\Phi(\beta)-\left\langle\theta\left(g^{-1}\right),\beta\right\rangle$, we differentiate it at $g=e$ using the property that:

$$\tilde{\Theta}(X,Y)=-\langle d\theta(X),Y\rangle\ ,\ X,Y\in\mathfrak{g}$$

- we obtain : $\langle Q,[\beta,Z]\rangle+\tilde{\Theta}(\beta,Z)=0$
- From last Souriau equation, if we use the identities $\beta=\frac{\partial S}{\partial Q}$, $ad_{\beta}Z=[\beta,Z]$ and $\tilde{\Theta}(\beta,Z)=\langle\Theta(\beta),Z\rangle$
- Then we can deduce that: $\left\langle ad^*_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right),Z\right\rangle=0,\ \ \forall Z$
- So, Entropy $S(Q)$ should verify:

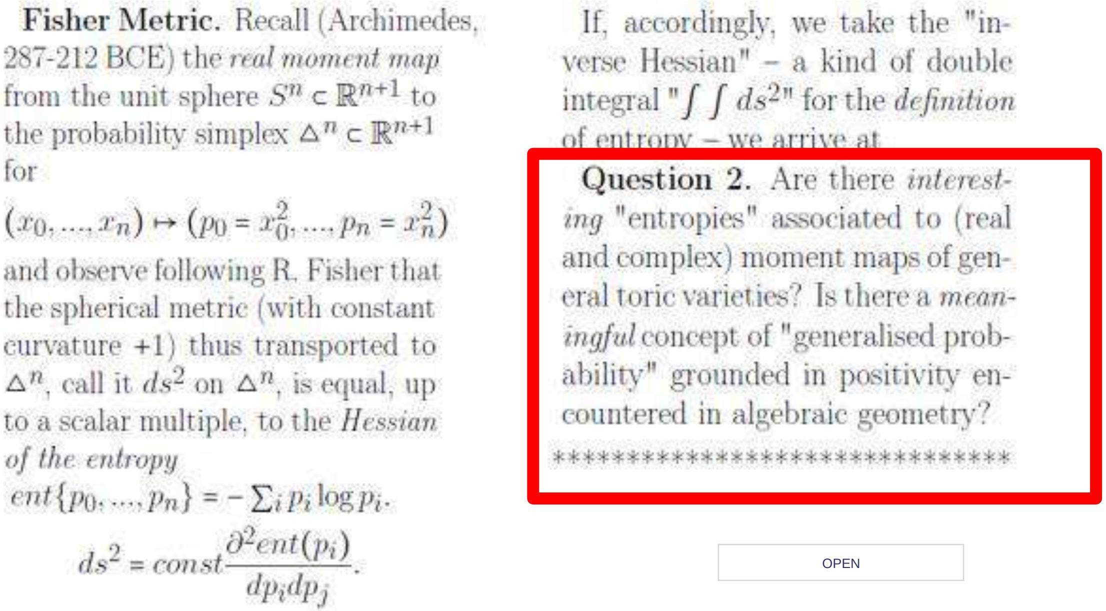
$$ad^*_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0 \qquad \{S,H\}_{\tilde{\Theta}}(Q)=0\ \ \forall H:\mathfrak{g}^*\to R,\ \ Q\in\mathfrak{g}^*$$

$$\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle Q,\left[\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right]\right\rangle+\tilde{\Theta}\left(\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right)=0$$

Gromov question: Are there « entropies » associated to moment maps

Bernoulli Lecture - What is Probability?

- 27 March 2018 - CIB - EPFL - Switzerland
- Lecturer: Mikhail Gromov
- https://bernoulli.epfl.ch/images/website/What_is_Probability_v2(2).mp4
- http://forum.cs-dc.org/uploads/files/1525172771489-alternative-probabilities-2018.pdf



Fisher Metric. Recall (Archimedes, 287-212 BCE) the *real moment map* from the unit sphere $S^n \subset \mathbb{R}^{n+1}$ to the probability simplex $\Delta^n \subset \mathbb{R}^{n+1}$ for

$$(x_0, \dots, x_n) \mapsto (p_0 = x_0^2, \dots, p_n = x_n^2)$$

and observe following R. Fisher that the spherical metric (with constant curvature +1) thus transported to Δ^n, call it ds^2 on Δ^n, is equal, up to a scalar multiple, to the *Hessian of the entropy*

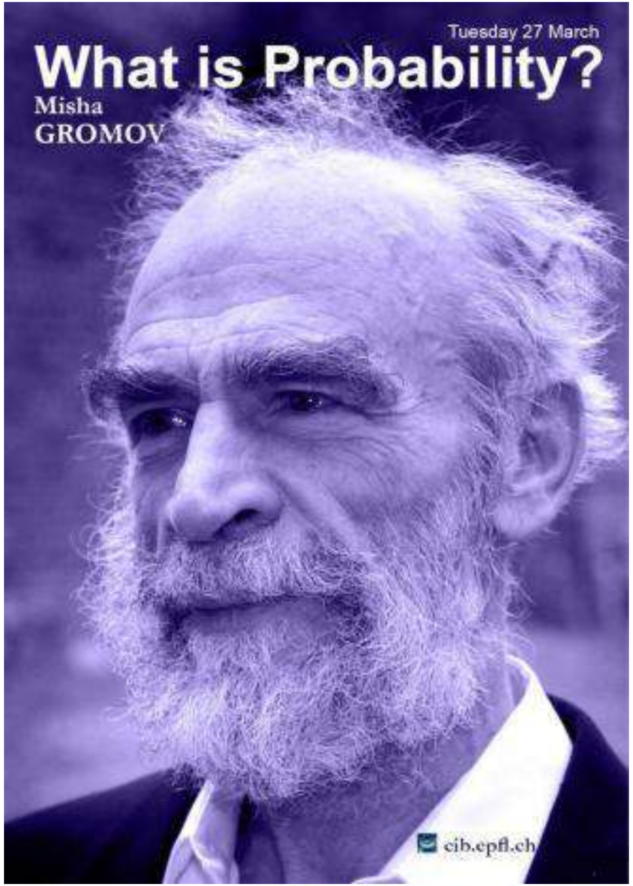
$$ent\{p_0, \dots, p_n\} = -\sum_i p_i \log p_i.$$

$$ds^2 = const \frac{\partial^2 ent(p_i)}{dp_i dp_j}.$$

If, accordingly, we take the "inverse Hessian" – a kind of double integral "$\int \int ds^2$" for the *definition* of entropy – we arrive at

Question 2. Are there *interesting* "entropies" associated to (real and complex) moment maps of general toric varieties? Is there a *meaningful* concept of "generalised probability" grounded in positivity encountered in algebraic geometry?

****



Entropy as Invariant Casimir Function in Coadjoint Representation

NEW GEOMETRIC DEFINITION OF ENTROPY

$$\{S,H\}_{\tilde{\Theta}}(Q)=0$$

$$ad^{*}_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0$$

$$\{S,H\}(Q)=\left\langle Q,\left[\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right]\right\rangle=-C_{ij}^{k}Q_{k}\frac{\partial S}{\partial Q_{i}}\cdot\frac{\partial H}{\partial Q_{j}}$$

$$\left[e_{i},e_{j}\right]=C_{ij}^{k}e_{k}\quad,\quad C_{ij}^{k}\ \text{ structure coefficients}$$

$$\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle Q,\left[\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right]\right\rangle+\tilde{\Theta}\left(\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right)=0\quad,\quad\forall H:\mathfrak{g}^{*}\rightarrow R,\ \ Q\in\mathfrak{g}^{*}$$

$$\tilde{\Theta}(X,Y)=J_{[X,Y]}-\{J_{X},J_{Y}\}\quad\text{where}\ \ J_{X}(x)=\langle J(x),X\rangle$$

$$\tilde{\Theta}(X,Y)=\langle\Theta(X),Y\rangle\ \text{ with }\ \Theta(X)=T_{e}\theta(X(e))$$

$$\theta(g)=Q\left(Ad_{g}(\beta)\right)-Ad_{g}^{*}(Q)$$

Souriau Entropy and Casimir Invariant Function

Demo

$$\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle Q,\left[\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right]\right\rangle+\left\langle \Theta\left(\frac{\partial S}{\partial Q}\right),\frac{\partial H}{\partial Q}\right\rangle=0$$

$$\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle Q,ad_{\frac{\partial S}{\partial Q}}\frac{\partial H}{\partial Q}\right\rangle+\left\langle \Theta\left(\frac{\partial S}{\partial Q}\right),\frac{\partial H}{\partial Q}\right\rangle=0$$

$$\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle ad^{*}_{\frac{\partial S}{\partial Q}}Q,\frac{\partial H}{\partial Q}\right\rangle+\left\langle \Theta\left(\frac{\partial S}{\partial Q}\right),\frac{\partial H}{\partial Q}\right\rangle=0$$

$$\forall H,\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle ad^{*}_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right),\frac{\partial H}{\partial Q}\right\rangle=0\Rightarrow ad^{*}_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0$$

Souriau Entropy and Casimir Invariant Function

Link with Souriau development

- Souriau property: $\beta \in Ker\tilde{\Theta}_{\beta} \Rightarrow \langle Q, [\beta, Z] \rangle + \tilde{\Theta}(\beta, Z) = 0$

$$\Rightarrow \langle Q, ad_{\beta}Z \rangle + \tilde{\Theta}(\beta, Z) = 0 \Rightarrow \langle ad_{\beta}^{*}Q, Z \rangle + \tilde{\Theta}(\beta, Z) = 0$$

$$\beta = \frac{\partial S}{\partial Q} \Rightarrow \left\langle ad_{\frac{\partial S}{\partial Q}}^{*} Q, Z \right\rangle + \tilde{\Theta}\left(\frac{\partial S}{\partial Q}, Z \right) = \left\langle ad_{\frac{\partial S}{\partial Q}}^{*} Q, Z \right\rangle + \left\langle \Theta\left(\frac{\partial S}{\partial Q} \right), Z \right\rangle = 0$$

$$\Rightarrow \left\langle ad_{\frac{\partial S}{\partial Q}}^{*} Q + \Theta\left(\frac{\partial S}{\partial Q} \right), Z \right\rangle = 0, \forall Z$$

$$\Rightarrow ad_{\frac{\partial S}{\partial Q}}^{*} Q + \Theta\left(\frac{\partial S}{\partial Q} \right) = 0$$

Souriau relation on foliation

- In his 1974 paper, Jean-Marie Souriau has written (without proof):
  $$\left\langle Q,[\beta,Z]\right\rangle+\tilde{\Theta}(\beta,Z)=0$$
- To prove this equation, we have to consider the parametrized curve
  $$t\mapsto Ad_{\exp(tZ)}\beta \text{ with } Z\in\mathfrak{g} \text{ and } t\in R$$
- The parameterized curve $Ad_{\exp(tZ)}\beta$ passes, for $t=0$, through the point β, since $Ad_{\exp(0)}$ is the identical map of the Lie Algebra $\mathfrak{g}$. This curve is in the adjoint orbit of β. So by taking its derivative with respect to t , then for $t=0$, we obtain a tangent vector in β at the adjoint orbit of this point. When Z takes all possible values in $\mathfrak{g}$, the vectors thus obtained generate all the vector space tangent in β to the orbit of this point:

$$\left.\frac{d\Phi\left(Ad_{\exp(tZ)}\beta\right)}{dt}\right|_{t=0}=\left\langle\frac{d\Phi}{d\beta},\left(\left.\frac{d\left(Ad_{\exp(tZ)}\beta\right)}{dt}\right|_{t=0}\right)\right\rangle=\left\langle Q,ad_Z\beta\right\rangle=\left\langle Q,[Z,\beta]\right\rangle$$

Souriau relation on foliation

- As we have seen before $\Phi\left(Ad_g\beta\right)=\Phi(\beta)-\left\langle\theta\left(g^{-1}\right),\beta\right\rangle$. If we set $g=\exp(tZ)$, we obtain:

$$\Phi\left(Ad_{\exp(tZ)}\beta\right)=\Phi(\beta)-\left\langle\theta\left(\exp(-tZ)\right),\beta\right\rangle$$

- By derivation with respect to t at $t=0$, we finally recover the equation given by Souriau :

$$\left.\frac{d\Phi\left(Ad_{\exp(tZ)}\beta\right)}{dt}\right|_{t=0}=\left\langle Q,[Z,\beta]\right\rangle=-\left\langle d\theta(-Z),\beta\right\rangle \text{ with } \tilde{\Theta}(X,Y)=-\left\langle d\theta(X),Y\right\rangle$$

Fundamental Equation of Geometric Thermodynamic: Entropy Function is an Invariant Casimir Function in Coadjoint Representation

Entropy S

$$S:\mathfrak{g}^* \rightarrow R$$
$$Q \mapsto S(Q)$$

Heat Q, (Planck) Temperature β and Φ Massieu Characteristic Function

$$S(Q)=\langle \beta, Q\rangle-\Phi(\beta)\ ,\ Q=\frac{\partial \Phi(\beta)}{\partial \beta}\in \mathfrak{g}^*\ \ ,\ \beta=\frac{\partial S(Q)}{\partial Q}\in \mathfrak{g}$$

Invariance of Entropy S Under the action of the Group

$$Q\left(Ad_g(\beta)\right)=Ad_g^*(Q)+\theta(g)$$

$$S\left(Q\left(Ad_g(\beta)\right)\right)=S(Q)$$

→

New Definition of Entropy S as Invariant Casimir Function in Coadjoint Representation

$$ad^*_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0$$

$$\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle Q,\left[\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right]\right\rangle+\tilde{\Theta}\left(\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right)=0$$

$$\Theta(X)=T_e\theta\left(X(e)\right)$$

$$\tilde{\Theta}(X,Y)=\langle \Theta(X),Y\rangle=J_{[X,Y]}-\{J_X,J_Y\}=-\langle d\theta(X),Y\rangle$$

Moment Map J

Lie Groups Thermodynamic Equations and its extension (1/3)

Q : Heat, element of dual Lie Algebra

β : (Planck) température element of Lie algebra

$$Q\left(Ad_g(\beta)\right)=Ad_g^*(Q)+\theta\left(g\right)$$

$\theta(g)$: Souriau Cocyle

Φ : Massieu Characteristic Function

J : Souriau Moment Map

$$\Phi(\beta)=-\log\int_M e^{-\langle\beta,J(\xi)\rangle}d\lambda$$

$$J:M\rightarrow\mathfrak{g}^*$$

S : Entropy

Legendre Transform

$$S(Q)=\langle\beta,Q\rangle-\Phi(\beta)\ \ \text{with}\ \ Q=\frac{\partial\Phi(\beta)}{\partial\beta}\in\mathfrak{g}^*\ \ \text{and}\ \ \beta=\frac{\partial S(Q)}{\partial Q}\in\mathfrak{g}$$

I : Fisher Information Metric

$$I(\beta)=-\frac{\partial^2\Phi}{\partial\beta^2}$$

Lie Groups Thermodynamic Equations and its extension (2/3)

Entropy Invariance under the action of the Group !

$$S\left(Ad_g^{\#}(Q)\right)=S\left(Q\right)$$

$$Ad_g^{\#}(Q)=Ad_g^{*}(Q)+\theta\left(g\right)$$

Souriau characteristic of the foliation

$$\left\langle Q,[\beta,Z]\right\rangle+\tilde{\Theta}\left(\beta,Z\right)=0$$

Entropy Solution of Casimir Equation

$$ad^{*}_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0$$

Souriau cocycle

$$\Theta(X)=T_e\theta\left(X(e)\right)$$

$$\theta\left(g\right)=Q\left(Ad_g(\beta)\right)-Ad_g^{*}(Q)$$

Entropy & Poisson Bracket →

$$\left\{S,H\right\}_{\tilde{\Theta}}\left(Q\right)=\left\langle Q,\left[\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right]\right\rangle+\tilde{\Theta}\left(\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right)=0$$

$$\tilde{\Theta}\left(X,Y\right)=\left\langle\Theta(X),Y\right\rangle=J_{[X,Y]}-\left\{J_X,J_Y\right\}=-\left\langle d\theta(X),Y\right\rangle$$

Lie Groups Thermodynamic Equations and its extension (3/3)

Entropy Production

$$dS = \tilde{\Theta}_{\beta}\left(\frac{\partial H}{\partial Q},\beta\right)dt$$

2nd principle is related to positivity of Fisher tensor

$$\frac{dS}{dt} = \tilde{\Theta}_{\beta}\left(\frac{\partial H}{\partial Q},\frac{\partial S}{\partial Q}\right) \geq 0$$

Metric Tensor related to Fisher Metric

$$\tilde{\Theta}_{\beta}\left(\frac{\partial H}{\partial Q},\beta\right) = \tilde{\Theta}\left(\frac{\partial H}{\partial Q},\beta\right) + \left\langle Q,\left[\frac{\partial H}{\partial Q},\beta\right]\right\rangle$$

Time Evolution of Heat wrt to Hamiltonian H

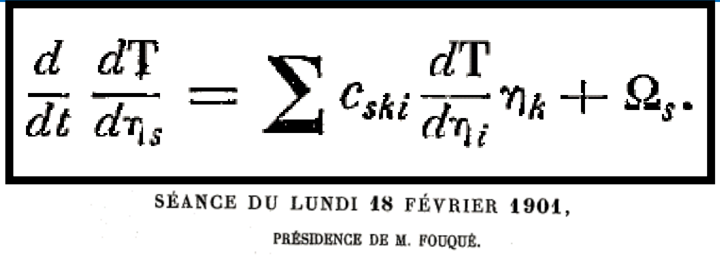
$$\frac{dQ}{dt} = ad^{*}_{\frac{\partial H}{\partial Q}}Q + \Theta\left(\frac{\partial H}{\partial Q}\right)$$

Stochastic Equation

$$dQ + \left[ad^{*}_{\frac{\partial H}{\partial Q}}Q + \Theta\left(\frac{\partial H}{\partial Q}\right)\right]dt + \sum_{i=1}^{N}\left[ad^{*}_{\frac{\partial H_i}{\partial Q}}Q + \Theta\left(\frac{\partial H_i}{\partial Q}\right)\right] \circ dW_i(t) = 0$$

EULER-POINCARE EQUATION … IN THERMODYNAMICS


Leonhard EULER



$$\boxed{\frac{d}{dt}\frac{dT}{d\eta_s} = \sum c_{ski}\frac{dT}{d\eta_i}\eta_k + \Omega_s.}$$

SÉANCE DU LUNDI 18 FÉVRIER 1901,

PRÉSIDENCE DE M. FOUQUÉ.


Henri POINCARE


François MASSIEU

$$Q\left(Ad_g(\beta)\right) = Ad_g^*(Q) + \theta\left(g\right)$$

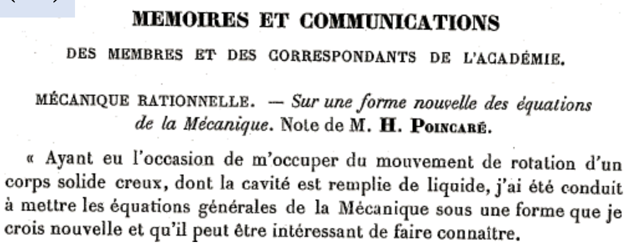
$$S(Q) = \langle \beta, Q \rangle - \Phi(\beta)$$

$$Q = \frac{\partial \Phi(\beta)}{\partial \beta} \quad , \beta = \frac{\partial S(Q)}{\partial Q}$$

$$\frac{dQ}{dt} = ad^*_{\frac{\partial H}{\partial Q}} Q + \Theta\left(\frac{\partial H}{\partial Q}\right)$$

$$ad^*_{\frac{\partial S}{\partial Q}} Q + \Theta\left(\frac{\partial S}{\partial Q}\right) = 0$$

Henri Poincaré
C.R. Acad. Sci. Paris, Tome CXXXII 7, 369–371 (1901)



MEMOIRES ET COMMUNICATIONS

DES MEMBRES ET DES CORRESPONDANTS DE L'ACADÉMIE.

MÉCANIQUE RATIONNELLE. — *Sur une forme nouvelle des équations de la Mécanique.* Note de M. H. POINCARÉ.

« Ayant eu l'occasion de m'occuper du mouvement de rotation d'un corps solide creux, dont la cavité est remplie de liquide, j'ai été conduit à mettre les équations générales de la Mécanique sous une forme que je crois nouvelle et qu'il peut être intéressant de faire connaître.

$Q \in \mathfrak{g}^*$: Geometric Heat

$\beta \in \mathfrak{g}$: Geometric Planck Temperature

$\Phi(\beta) : \mathfrak{g} \to \mathbb{R}$ Massieu Potential

$S(Q) : \mathfrak{g}^* \to \mathbb{R}$ Entropy

$$\boxed{\frac{d}{dt}\frac{\partial \Phi}{\partial \beta} = ad^*_{\frac{\partial H}{\partial Q}} \frac{\partial \Phi}{\partial \beta} + \Theta\left(\frac{\partial H}{\partial Q}\right)}$$

$$\boxed{\left\{S, H\right\}_{\tilde{\Theta}}\left(Q\right) = 0}$$

Entropy = Casimir Function on Symplectic Foliation (generated by Coadjoint Orbits via moment map)

https://www.esi.ac.at/events/t2213/

Euler-Poincaré Equation in case of Non-Null Cohomology

$$\frac{dQ}{dt} = ad^*_{\frac{\partial H}{\partial Q}} Q + \Theta\left(\frac{\partial H}{\partial Q}\right)$$

$$Q = \frac{\partial \Phi}{\partial \beta}$$

$$\frac{d}{dt}\frac{\partial \Phi}{\partial \beta} = ad^*_{\frac{\partial H}{\partial Q}} \frac{\partial \Phi}{\partial \beta} + \Theta\left(\frac{\partial H}{\partial Q}\right)$$

$$\left(ad^*_{\frac{\partial H}{\partial Q}}\frac{\partial \Phi}{\partial \beta}\right)_j + \Theta\left(\frac{\partial H}{\partial Q}\right)_j = C^k_{ij}\, ad^*_{\left(\frac{\partial H}{\partial Q}\right)^i}\left(\frac{\partial \Phi}{\partial \beta}\right)_k + \Theta_j$$

de Saxcé, G. Euler-Poincaré equation for Lie groups with non null symplectic cohomology. Application to the mechanics. In GSI 2019. LNCS; Nielsen, F., Barbaresco, F., Eds.; Springer: Berlin, Germany, 2019; Volume 11712

« Ayant eu l'occasion de m'occuper du mouvement de rotation d'un corps solide creux, dont la cavité est remplie de liquide, j'ai été conduit à mettre les équations générales de la mécanique sous une forme que je crois nouvelle et qu'il peut être intéressant de faire connaître » - Henri Poincaré, CRAS, 18 Février 1901

SÉANCE DU LUNDI 18 FÉVRIER 1901,

PRÉSIDENCE DE M. FOUQUÉ.

MEMOIRES ET COMMUNICATIONS

DES MEMBRES ET DES CORRESPONDANTS DE L'ACADÉMIE.

MÉCANIQUE RATIONNELLE. — *Sur une forme nouvelle des équations de la Mécanique*. Note de M. H. POINCARÉ.

« Ayant eu l'occasion de m'occuper du mouvement de rotation d'un corps solide creux, dont la cavité est remplie de liquide, j'ai été conduit à mettre les équations générales de la Mécanique sous une forme que je crois nouvelle et qu'il peut être intéressant de faire connaître.

$$\frac{d}{dt}\frac{dT}{d\eta_s} = \sum c_{ski}\frac{dT}{d\eta_i}\eta_k + \Omega_s.$$

« Elles sont surtout intéressantes dans le cas où U étant nul, T ne dépend que des η » - Henri Poincaré

Koszul Poisson Cohomology and Entropy Characterization

- Poisson Cohomology was introduced by A. Lichnerowicz and J.L. Koszul.

- Koszul Cohomology and seminal work of Elie Cartan. Koszul made reference to seminal E. Cartan paper

  - “***Elie Cartan does not explicitly mention $\Lambda(g')$ [the complex of alternate forms on a Lie algebra], because he treats groups as symmetrical spaces and is therefore interested in differential forms which are invariant to both by the translations to the left and the translations to the right, which corresponds to the elements of $\Lambda(g')$ invariant by the prolongation of the coadjoint representation. Nevertheless, it can be said that by 1929 an essential piece of the cohomological theory of Lie algebras was in place.***” – Jean-Louis Koszul

THALES

Koszul Poisson Cohomology and Entropy Characterization

- Y. Vorob’ev and M.V. Karasev have suggested cohomology classification in terms of closed forms and de Rham Cohomology of coadjoint orbits Ω (called Euler orbits by authors), symplectic leaves of a Poisson manifold N .
- Let $Z^k(\Omega)$ and $H^k(\Omega)$ be the space of closed k-forms on Ω and their de Rham cohomology classes.
- Considering the base of the fibration of N by these orbits as N/Ω , they have introduced the smooth mapping

$Z^k[\Omega]=C^\infty\left(N/\Omega\to Z^k(\Omega)\right)$ and $H^k[\Omega]=C^\infty\left(N/\Omega\to H^k(\Omega)\right)$

- The elements of $Z^k(\Omega)$ are closed forms on Ω , depending on coordinates on N/Ω
- Then $H^0[\Omega]=Casim(N)$ is the set of Casimir functions on N , of functions which are constant on all Euler orbits.
- **Entropy is then characterized by zero-dimensional de Rham Cohomology**.
- The center of Poisson algebra induced from the symplectic structure is the zero-dimensional de Rham cohomology group, the Casimir functions.

Poincaré Unit Disk and SU(1,1) Lie Group

- The group of complex unimodular pseudo-unitary matrices $SU(1,1)$:

$$G = SU(1,1) = \left\{ \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} / |a|^2 - |b|^2 = 1,\ a,b \in C \right\}$$

- the Lie algebra $\mathfrak{g} = \mathfrak{su}(1,1)$ is given by:

$$\mathfrak{g} = \left\{ \begin{pmatrix} -ir & \eta \\ \eta^* & ir \end{pmatrix} / r \in R, \eta \in C \right\}$$

with the following bases $(u_1, u_2, u_3) \in \mathfrak{g}$:

$$u_1 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ u_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ u_3 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

with the commutation relation:

$$[u_3, u_2] = u_1, [u_3, u_1] = u_2, [u_2, u_1] = -u_3$$

Poincaré Unit Disk and SU(1,1) Lie Group

- Dual base on dual Lie algebra is named
  $\left(u_1^*, u_2^*, u_3^*\right) \in \mathfrak{g}^*$
- The dual vector space $\mathfrak{g}^* = \mathfrak{su}^*(1,1)$ can be identified with the subspace of $\mathfrak{sl}(2,C)$ of the form:

$$\mathfrak{g}^* = \left\{ \begin{pmatrix} z & x+iy \\ -x+iy & -z \end{pmatrix} = x\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} / \, x, y, z \in R \right\}$$

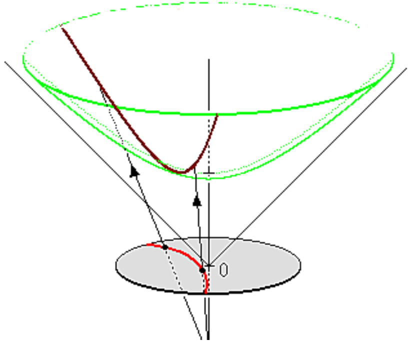
- Coadjoint action of $g \in G$ on dual Lie algebra $\xi \in \mathfrak{g}^*$ is written $g.\xi$

Coadjoint Orbit of SU(1,1) and Souriau Moment Map

$$J(z) = r\left(\frac{z+z^*}{\left(1-|z|^2\right)} u_1^* + \frac{z-z^*}{i\left(1-|z|^2\right)} u_2^* + \frac{1+|z|^2}{\left(1-|z|^2\right)} u_3^* \right) \in \mathrm{O}\left(r u_3^*\right), z \in D$$

- J is linked to the natural action of G on D (by fractional linear transforms) but also the coadjoint action of G on $\mathrm{O}\left(r u_3^*\right) = G / K$
- J^{-1} could be interpreted as the stereographic projection from the two-sphere S^2 onto $C \cup \infty$:

The coadjoint action of $G = SU(1,1)$ is the upper sheet $x_3 > 0$ of the two-sheet hyperboloid



Charles-Michel Marle, Projection stéréographique et moments, hal-02157930, version 1, Juin 2019

$$\left\{ \xi = x_1 u_1^* + x_2 u_2^* + x_3 u_3^* : -x_1^2 - x_2^2 + x_3^2 = r^2 \right\}$$

Moment Map for SU(1,1)

Invariant Moment Map

- The associated moment map $J : D \rightarrow su^*(1,1)$ defined by $J(z).u_i = J_i(z,z^*)$, maps D into a coadjoint orbit in $su^*(1,1)$.
- Then, we can write the moment map as a matrix element of $su^*(1,1)$:

$$J(z) = J_1\left(z,z^*\right)u_1^* + J_2\left(z,z^*\right)u_2^* + J_3\left(z,z^*\right)u_3^*$$

$$J(z) = r\left(\frac{z+z^*}{\left(1-|z|^2\right)}u_1^* + \frac{z-z^*}{i\left(1-|z|^2\right)}u_2^* + \frac{1+|z|^2}{\left(1-|z|^2\right)}u_3^*\right), z \in D$$

$$J(z) = r\begin{pmatrix} \frac{1+|z|^2}{1-|z|^2} & -2\frac{z^*}{1-|z|^2} \\ 2\frac{z}{1-|z|^2} & -\frac{1+|z|^2}{1-|z|^2} \end{pmatrix} \in \mathfrak{g}^*$$

$$\mathfrak{g}^* = \left\{\begin{pmatrix} z & x+iy \\ -x+iy & -z \end{pmatrix} = x\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} / x,y,z \in R\right\}$$

Poisson Bracket for SU(1,1)

- Since the unit disk is Kählerian, it is symplectic and so can be given a phase space structure and interpretation. This Poisson Bracket could be written in terms of the Poincare disk coordinates as:

$$\{f,g\} = \frac{\left(1-|z|^2\right)^2}{2i}\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial z^*}-\frac{\partial f}{\partial z^*}\frac{\partial g}{\partial z}\right)$$

- It is possible to define new coordinates (q,p) that are canonical in the sense that:

$$\{f,g\} = \left(\frac{\partial f}{\partial q}\frac{\partial g}{\partial p}-\frac{\partial f}{\partial p}\frac{\partial g}{\partial q}\right)$$ with coordinates given by: $$\frac{q+ip}{2}=\frac{z}{\sqrt{1-|z|^2}}$$

- The Metriplectic equation is then given by:

$$\frac{\partial f}{\partial t}=\{f,H\}+(f,H)=\frac{\left(1-|z|^2\right)^2}{2i}\left(\frac{\partial f}{\partial z}\frac{\partial H}{\partial z^*}-\frac{\partial f}{\partial z^*}\frac{\partial H}{\partial z}\right)+(f,H)$$

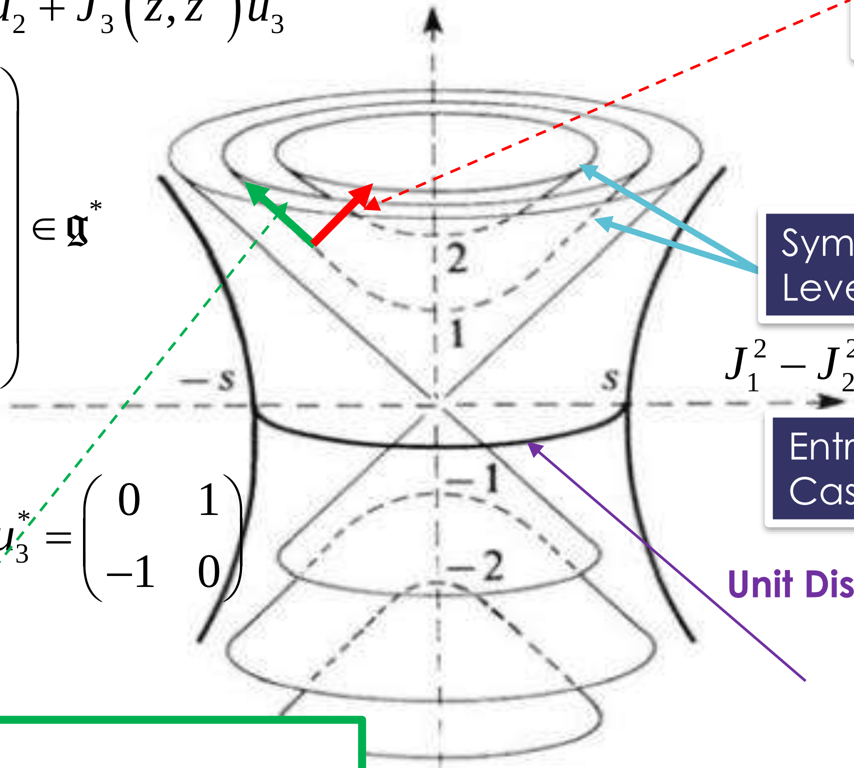
Souriau Lie Groups Thermodynamics Model for SU(1,1) and Unit Disk

$$J(z)=J_1\left(z,z^*\right)u_1^*+J_2\left(z,z^*\right)u_2^*+J_3\left(z,z^*\right)u_3^*$$

$$J(z)=\rho\begin{pmatrix}\dfrac{1+|z|^2}{1-|z|^2} & -2\dfrac{z^*}{1-|z|^2}\\ 2\dfrac{z}{1-|z|^2} & -\dfrac{1+|z|^2}{1-|z|^2}\end{pmatrix}\in\mathfrak{g}^*$$

$$J_1u_1^*+J_2u_2^*+J_3u_3^*\in su^*(1,1)$$

$$u_1^*=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\ u_2^*=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix},\ u_3^*=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$



Dissipative flow (transverse to symplectic leaves)

$$\boxed{\frac{\partial f}{\partial t}=(f,H)}$$

Symplectic Leaves = Level sets of Entropy

$$J_1^2-J_2^2-J_3^2=\rho^2\quad,\quad J_1\geq\rho$$

Entropy = Invariant Casimir Function

Unit Disk as Symplectic Manifold

$$D=\left\{z\in\mathbb{C}\,/\,|z|<1\right\}$$

$$\omega_\rho=2ir\frac{dz\wedge dz^*}{\left(1-|z|^2\right)^2}$$

Non Dissipative flow (along the symplectic leaves)

$$\boxed{\frac{\partial f}{\partial t}=\{f,H\}=\frac{\left(1-|z|^2\right)^2}{2i}\left(\frac{\partial f}{\partial z}\frac{\partial H}{\partial z^*}-\frac{\partial f}{\partial z^*}\frac{\partial H}{\partial z}\right)}$$

Souriau Gibbs density for SU(1,1)

Covariant Gibbs density

- We can the write the covariant Gibbs density in the unit disk given by moment map of the Lie group $SU(1,1)$ and geometric temperature in its Lie algebra $\beta \in \Lambda_\beta$:

$$p_{Gibbs}(z) = \frac{e^{-\langle J(z),\beta\rangle}}{\int\limits_D e^{-\langle J(z),\beta\rangle} d\lambda(z)} \text{ with } d\lambda(z) = 2ir\frac{dz \wedge dz^*}{\left(1-|z|^2\right)^2}$$

$$p_{Gibbs}(z) = \frac{e^{-\left\langle r\left(2\mathfrak{I}bb^+ - Tr\left(\mathfrak{I}bb^+\right)I\right),\beta\right\rangle}}{\int\limits_D e^{-\langle J(z),\beta\rangle} d\lambda(z)} = \frac{e^{-\left\langle r\begin{pmatrix} \frac{1+|z|^2}{\left(1-|z|^2\right)} & \frac{-2z^*}{\left(1-|z|^2\right)} \\ \frac{2z}{\left(1-|z|^2\right)} & -\frac{1+|z|^2}{\left(1-|z|^2\right)} \end{pmatrix}, \begin{pmatrix} ir & \eta \\ \eta^* & -ir \end{pmatrix}\right\rangle}}{\int\limits_D e^{-\langle J(z),\beta\rangle} d\lambda(z)}$$

$$J(z) = r\left(2Mbb^+ - Tr\left(Mbb^+\right)I\right) \text{ with } M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } b = \frac{1}{1-|z|^2}\begin{bmatrix} 1 \\ -z \end{bmatrix}$$

Souriau Gibbs density

Covariant Gibbs Density

$$p_{Gibbs}(z)=\frac{e^{-\left\langle \rho\begin{pmatrix}\frac{1+|z|^2}{\left(1-|z|^2\right)} & \frac{-2z^*}{\left(1-|z|^2\right)}\\ \frac{2z}{\left(1-|z|^2\right)} & -\frac{1+|z|^2}{\left(1-|z|^2\right)}\end{pmatrix},\begin{pmatrix} ir & \eta \\ \eta^* & -ir\end{pmatrix}\right\rangle}}{\int\limits_D e^{-\langle J(z),\beta\rangle}d\lambda(z)}$$

- To write the Gibbs density with respect to its statistical moments, we have to express the density with respect to $Q=E\left[J(z)\right]$
- Then, we have to invert the relation between Q and β, to replace $\beta=\begin{pmatrix} ir & \eta \\ \eta^* & -ir\end{pmatrix}\in\Lambda_\beta$ by $\beta=\Theta^{-1}(Q)\in\mathfrak{g}$ where $Q=\frac{\partial\Phi(\beta)}{\partial\beta}=\Theta(\beta)\in\mathfrak{g}^*$ with $\Phi(\beta)=-\log\int\limits_D e^{-\langle J(z),\beta\rangle}d\lambda(z)$ deduce from Legendre tranform. The mean moment map is given by:

$$Q=E\left[J(z)\right]=E\left[\rho\begin{pmatrix}\frac{1+|w|^2}{\left(1-|w|^2\right)} & \frac{-2w^*}{\left(1-|w|^2\right)}\\ \frac{2w}{\left(1-|w|^2\right)} & -\frac{1+|w|^2}{\left(1-|w|^2\right)}\end{pmatrix}\right]\quad \text{where}\quad w\in D$$

THALES

Lars Onsager & Baptiste Coquinot: *Symplectic Foliation Model of METRIPLECTIC FLOW*

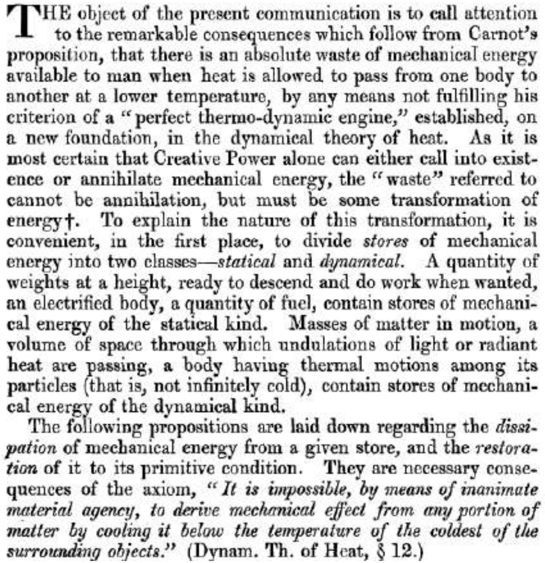

L. ONSAGER


B. COQUINOT

DISSIPATION CONCEPT INTRODUCTION BY LORD KELVIN

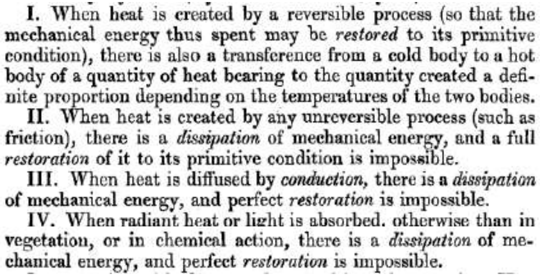
The concept of dissipation was introduced in the field of thermodynamics by Lord Kelvin (William Thomson) in 1852. Lord Kelvin deduced that **irreversible dissipative processes** will occur unless a process is governed by a "**perfect thermodynamic engine**". The processes that Lord Kelvin identified were **friction, diffusion, conduction of heat and the absorption of light**.

W. Thomson, On the universal tendency in nature to the dissipation of mechanical energy, Philosophical Magazine, Ser. 4, p. 304 (1852)



THE object of the present communication is to call attention to the remarkable consequences which follow from Carnot's proposition, that there is an absolute waste of mechanical energy available to man when heat is allowed to pass from one body to another at a lower temperature, by any means not fulfilling his criterion of a "perfect thermo-dynamic engine," established, on a new foundation, in the dynamical theory of heat. As it is most certain that Creative Power alone can either call into existence or annihilate mechanical energy, the "waste" referred to cannot be annihilation, but must be some transformation of energy†. To explain the nature of this transformation, it is convenient, in the first place, to divide *stores* of mechanical energy into two classes—*statical* and *dynamical*. A quantity of weights at a height, ready to descend and do work when wanted, an electrified body, a quantity of fuel, contain stores of mechanical energy of the statical kind. Masses of matter in motion, a volume of space through which undulations of light or radiant heat are passing, a body having thermal motions among its particles (that is, not infinitely cold), contain stores of mechanical energy of the dynamical kind.

The following propositions are laid down regarding the *dissipation* of mechanical energy from a given store, and the *restoration* of it to its primitive condition. They are necessary consequences of the axiom, "*It is impossible, by means of inanimate material agency, to derive mechanical effect from any portion of matter by cooling it below the temperature of the coldest of the surrounding objects.*" (Dynam. Th. of Heat, § 12.)



I. When heat is created by a reversible process (so that the mechanical energy thus spent may be *restored* to its primitive condition), there is also a transference from a cold body to a hot body of a quantity of heat bearing to the quantity created a definite proportion depending on the temperatures of the two bodies.

II. When heat is created by any unreversible process (such as friction), there is a *dissipation* of mechanical energy, and a full *restoration* of it to its primitive condition is impossible.

III. When heat is diffused by *conduction*, there is a *dissipation* of mechanical energy, and perfect *restoration* is impossible.

IV. When radiant heat or light is absorbed, otherwise than in vegetation, or in chemical action, there is a *dissipation* of mechanical energy, and perfect *restoration* is impossible.

OPEN

> **Mechanical Dissipation**

- Viscous Friction (Fluid Mechanics)
- Solid Friction (Dry or Coulomb Friction)
- Plastic Deformation
- Acoustic Damping (due to viscosity or thermal conduction)

> **Electromagnetic Dissipation**

- Ohmic or Joule Heating
- Eddy Current Losses (Induced currents in conductive materials)
- Dielectric Losses (imperfect polarization)
- Radiation Damping (accelerating charges emit radiation)

> **Thermal Dissipation**

- Heat Conduction
- Thermal Diffusion (Soret Effect)

> **Diffusion and Transport Processes**

- Mass Diffusion (molecular motion)
- Viscous Diffusion
- Electrical Conduction

> **Relaxation Phenomena**

- Magnetic relaxation (spin-lattice relaxation)

METRIPLECTIC FRAMEWORK FOR DISSIPATIVE HEAT EQUATION

METRIPLECTIC & GENERIC Models

- Systems that preserve energy throughout the phase are characterized by an Hamiltonian formulation of dynamics.
- Classical Hamiltonian systems cannot take into account dissipative effects, as irreversible changes from a thermodynamic standpoint (dissipative dynamics that do not preserve energy).
- A. N. Kaufman and P.J. Morrison have introduced in 1983, the **metriplectic** bracket by introducing a bracket formalism that ensures both conservation of energy and non-decrease of entropy, and that reduces to the standard Poisson bracket formalism in the limit of no dissipation.
- This model has been axiomatized in parallel by Grmela and his collaborators (**GENERIC method: General Equation for Non-Equilibrium Reversible Irreversible Coupling**).

METRIPLECTIC FRAMEWORK FOR DISSIPATIVE HEAT EQUATION

Metriplectic Model

- Dissipation could take two forms:
  - viscosity removes energy from the system (e.g. Navier-Stokes equation)
  - thermal diffusion with conservation of energy and entropy production by heat transfer (e.g. Boltzmann operator, Transport equations with collision operators,…).
- Metriplectic dynamics includes these kind of systems compliant both with the first and second thermodynamics principles.
- In the metriplectic formalism, evolution equation is given by a new bracket: $\{\{.,.\}\}$

$$\frac{df}{dt}=\{\{f,F\}\}=\underbrace{\{f,F\}}_{\substack{\textbf{Non-Dissipative Bracket}\\ \text{(Poisson Bracket)}}}+\underbrace{(f,F)}_{\substack{\textbf{Dissipative Bracket}\\ \text{(Metric Flow Bracket)}}}$$

METRIPLECTIC FRAMEWORK FOR DISSIPATIVE HEAT EQUATION

Metriplectic Model

$$\frac{df}{dt}=\{\{f,F\}\}=\{f,F\}+(f,F)$$

- Hamiltonian components is introduced by requiring:

$$F=H+S$$

- The 2nd bracket has 2 constraints :

$$(f,F)=(F,f) \text{ and } (f,f)\geq 0$$

- with **the entropy S selected from the set of Casimir invariants of the non-canonical Poisson bracket**, playing the role of a Lyapunov functional.
- A metriplectic vector field induced by F is given by the dynamics:

$$\frac{dz_i}{dt}=J_{ij}\frac{\partial F}{\partial z_j}+M_{ij}\frac{\partial F}{\partial z_j}$$

Metriplectic Model Compliance to Thermodynamics Principles

Metriplectic Model compliance with two first principles of thermodynamics

- First principle: **Energy conservation**

$$F = H + S$$

$$\frac{dH}{dt} = \{H,F\} + (H,F) = \{H,H\} + \{H,S\} + (H,H) + (H,S) = 0$$

$$\text{because } \begin{cases} \{H,H\} = 0 \text{ by symmetry} \\ \{f,S\} = 0, \quad \forall f \\ (H,f) = 0, \quad \forall f \end{cases}$$

Metriplectic Model Compliance to Thermodynamics Principles

Metriplectic Model compliance with two first principles of thermodynamics

- Second principle: **Entropy production**

$$\frac{dS}{dt}=\{S,F\}+(S,F)=0+(S,H)+(S,S)=(S,S)\geq 0$$

$$\text{because }\begin{cases}\{S,f\}=0,\quad \forall f\quad \text{(Casimir property)}\\ (f,H)=0,\quad \forall f\\ \text{positive semi-definite }\Rightarrow\quad (S,S)\geq 0\end{cases}$$

- The choice of thermal equilibrium is induced by selecting **Entropy as Casimir invariant function**.

Metriplectic Model and Casimir invariants

Metriplectic Model

- Finally in metriplectic systems, the geometry is determined by two compatible brackets, a Poisson bracket and a symmetric bracket:

$$\frac{df}{dt} = \{\{f,F\}\} = \{f,H\} + (f,S)$$

- The energy H is a Casimir invariant of the dissipative bracket, and the entropy S is a Casimir invariant of the Poisson bracket:

$$\{S,H\} = 0 \quad \forall H$$

$$(H,S) = 0 \quad \forall S$$

Metriplectic Flow on Symplectic Foliation & Transverse Metric Foliation

Foliation Leaves =
Level Sets of Energy

Foliation Leaves =
Level Sets of Entropy

METRIC FOLIATION $\mathfrak{I}^{\perp} \perp \mathfrak{I}$ **SYMPLECTIC FOLIATION**

1st Principle of Thermodynamics Preservation of Energy

2nd Principle of Thermodynamics Entropy Production

METRIPLECTIC FLOW

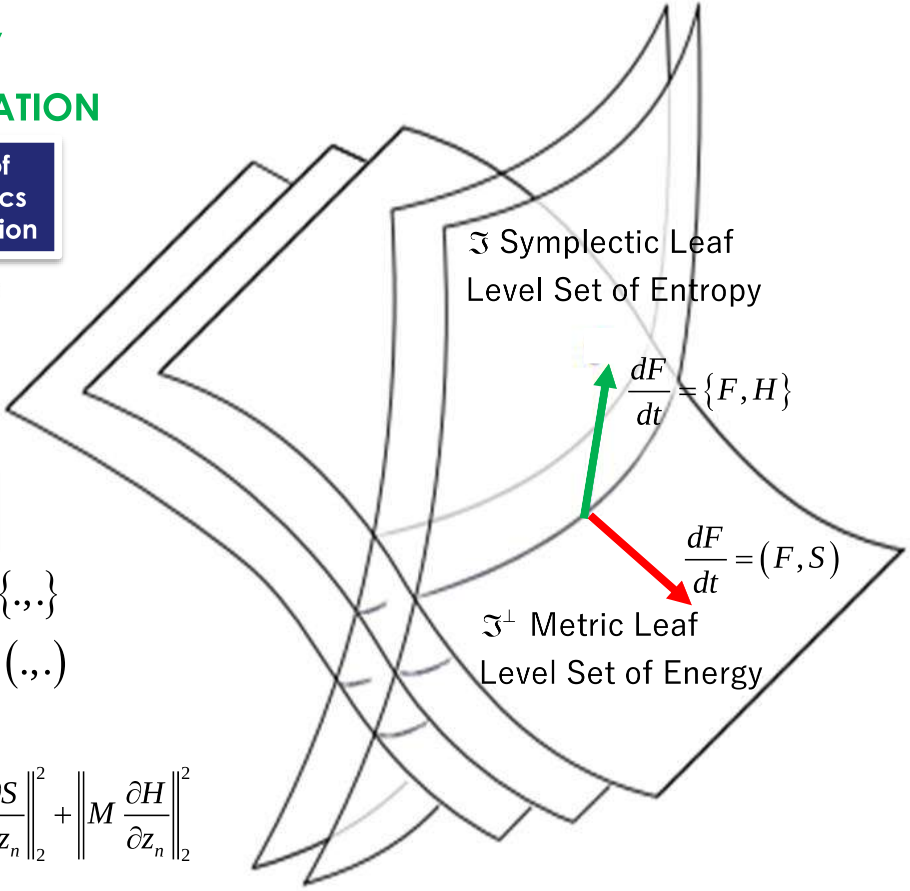
$$\frac{dF}{dt} = \{F,H\} + (F,S)$$

Non-dissipative Entropy = Constant

Dissipative Energy = Constant

$\{F,S\} = 0 \quad \forall F : S$ Entropy Casimir Function for $\{.,.\}$

$(F,H) = 0 \quad \forall F : H$ Energy Casimir Function for $(.,.)$



Thermodynamics-Informed Neural Networks

$$\Lambda = \frac{1}{N_{batch}} \sum_{n=0}^{N_{batch}} \left(\lambda \Lambda_n^{data} + \Lambda_n^{deg} \right) \qquad \Lambda_n^{data} = \left\| \frac{dz^{GT}}{dt} - \frac{dz^{net}}{dt} \right\|_2^2 \qquad \Lambda_n^{deg} = \left\| L \frac{\partial S}{\partial z_n} \right\|_2^2 + \left\| M \frac{\partial H}{\partial z_n} \right\|_2^2$$

SINN & MINN: Symplectic-Informed NN & Metriplectic-Informed NN

Symplectic Integrator

- Capture of symmetries (Geometry-Informed)
- Capture Noether Invariants via Souriau's Moment Map as Energy, Angular Momentum (Physics-Informed)

$$\frac{dF}{dt} = \{F, H\}$$

non-dissipative

(*) The time evolution of Hamilton's equations

$$\dot{p} = -\frac{\partial H}{\partial q} \quad , \quad \dot{q} = \frac{\partial H}{\partial p}$$

is a symplectomorphism, meaning that it conserves the symplectic 2-form $dp \wedge dq$. A numerical scheme is a symplectic integrator if it also conserves this 2-form and equation are reduce to:

$$z = (p, q) \Rightarrow \dot{z} = \{z, H(z)\}$$

Metriplectic Integrator

- Capture 1st principle (Energy preservation, Moment map)
- Capture 2nd principle (Entropy Production)

$$\frac{dF}{dt} = \{F, H\} + (F, S)$$

dissipative

$$F = H + S$$

(*) 1st principle: Preservation of Energy

$$\frac{dH}{dt} = 0$$

2nd Principle Entropy production

$$\frac{dS}{dt} \geq 0$$

Symplectic potentials and Guillemin Metric

- We consider **Action-Angle coordinates**: $\omega = dx^i \wedge d\theta_i$
- We also consider Moment Map $\mu : M \to \mathfrak{g}^*$ where $\left(x^1, \cdots, x^n\right)$ are coordinates on $\mathfrak{g}^*$ given by $x^i = \langle X_i, . \rangle$ where $\left(X_1, X_2, ..., X_n\right)$ is a base of vectors field of group action: $dx^i = -i_{X_i}\omega$. We can select angular coordinates such that $X_i = \frac{\partial}{\partial \theta_i}$
- Symplectic coordinates and complex structure: Consider a complexe structure J

$$Jdx^i = G^{ij} d\theta_j \quad \text{and} \quad Jd\theta_i = -G_{ij} dx^j \quad \text{where} \quad G_{ij} = \left(G^{ij}\right)^{-1}$$

- We can then deduce the metric:

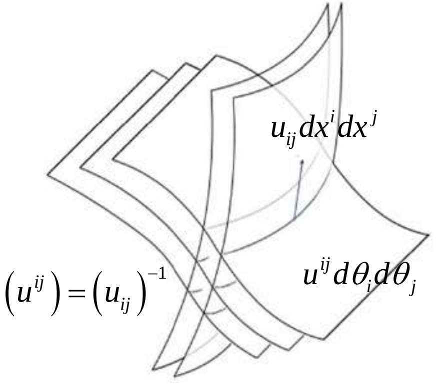
$$g = G_{ij} dx^i dx^j + G^{ij} d\theta_i d\theta_j \quad \text{where} \quad \left(G_{ij}\right) \text{ is symetric positive definite}$$

- We can make appear a symplectic potential:

$$dJd\theta_i = -\frac{\partial G_{ij}}{\partial x^k} dx^k \wedge dx^j \text{ of type } (1,1)$$

$$dJd\theta_i = 0 \Rightarrow \frac{\partial G_{ij}}{\partial x^k} = \frac{\partial G_{ik}}{\partial x^j}, \exists u \text{ convex, } G_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j} = u_{ij}$$

$$g = u_{ij} dx^i dx^j + u^{ij} d\theta_i d\theta_j \quad \text{where} \quad \left(u^{ij}\right) = \left(u_{ij}\right)^{-1}$$



Transversally oriented foliation & the Godbillon–Vey characteristic class

GODBILLON–VEY CLASSES OF SYMPLECTIC FOLIATIONS by Kentaro Mikami

- Each transversally oriented foliation has the Godbillon–Vey characteristic class, and regular Poisson structures define symplectic foliations.
- K. Mikami has given a new interpretation and the explicit formula for a representative of the Godbillon–Vey characteristic classes of symplectic foliations in the context of Poisson geometry.
- For each transversely oriented foliation, we have the famous Godbillon-Vey characteristic class. When the symplectic foliations of regular Poisson structures are transversely oriented, they have the Godbillon-Vey characteristic classes. K. Mikami has given a formula defining their Godbillon-Vey classes in terms of Poisson structure.
- Transversely oriented foliations have secondary characteristic classes which are called the Godbillon-Vey class.
  - Reference: **J.-L. Koszul, Crochet de Schouten-Nijenhuis et cohomologie, in 'Elie Cartan et les Mathématiques d'aujourd'hui', 257-271; Société Math. de France, Astérisque, hors série, 1985.**

METRIPLECTIC MODEL

Metriplectic Model

- In coordinate realization, with a coordinate chart $\left(z_i\right)$, the Poisson bivector is represented by a set of coefficient function determining the Poisson bracket:

$$\{f,h\} = J_{ij}\frac{\partial f}{\partial z_i}\frac{\partial h}{\partial z_j} \Rightarrow \frac{dz_i}{dt} = J_{ij}\frac{\partial h}{\partial z_j}$$

- **For Lie-Poisson structure defined on the dual of a finite dimensional Lie algebra, we can introduce structure constants** with an N dimensional Lie algebra admitting a basis $\left\{e_1,\ldots,e_N\right\}$: $\left[e_i,e_j\right] = c_{ij}^k e_k$ (with summation convention over the repeated indices).
- The Lie-Poisson dynamics is given by:

$$J_{ij} = c_{ij}^k z_k \Rightarrow \frac{dz_j}{dt} = c_{ij}^k z_k \frac{\partial h}{\partial z_i}$$

METRIPLECTIC MODEL

Dissipation bracket as double bracket

- The double bracket is given by:

$$(f,h)=\sum_j J_{ij}J_{lj}\frac{\partial f}{\partial z_i}\frac{\partial h}{\partial z_l}=\sum_j c_{ij}^k c_{lj}^r z_k z_r \frac{\partial f}{\partial z_i}\frac{\partial f}{\partial z_l}$$

- with the metriplectic dynamics :

$$\frac{dz_j}{dt}=c_{ij}^k z_k \frac{\partial h}{\partial z_i}+\sum_i c_{ji}^k c_{li}^r z_r z_n \frac{\partial h}{\partial z_l}$$

METRIPLECTIC MODEL

The symmetric dissipative metriplectic bracket as Euclidean metric tensor on the symplectic leaves foliated by the Casimir invariants

- *Sato observed that the canonical form of the symmetric dissipative part of the metriplectic bracket is identified in terms of a 'canonical metric tensor' corresponding to an Euclidean metric tensor on the symplectic leaves foliated by the Casimir invariants.*
- It makes the link with Symplectic model of Lie groups Thermodynamics. A single generating Massieu function $\Phi = \langle \beta, Q \rangle - S$ **is sufficient to generate the dynamics by the action of the metriplectic bracket:**

$$\frac{dF}{dt} = \{\{F,\Phi\}\} = \{F,Q\} + (F,S) = \beta^{-1}\{F,\Phi\} - (F,\Phi)$$

$$\{F,S\} = 0 \text{ and } (F,Q) = 0$$

$$H = W + Q$$

A Single Generating Function Generates the Metriplectic Dynamics:
François Massieu Characteristic Function

Variational Principles of Thermodynamics

Variational Principles

- **D'Alembert's variational principles** and Gauss' principle of the least constraint, which are differential principles
- **Maupertuis' principle of the least action** and the Hamilton principle, which are integral principles

Thermodynamics can be embraced with variational principles

- **ONSAGER's principle of least dissipation of energy** restricted to the particular case of heat conduction in anisotropic continua
- **ONSAGER & MACHLUP's principle and TISZA & MANNING's principle of least dissipation of energy** extended for the case of adiabatically isolated, non-continuous systems.
- **PRIGOGINE's principle of minimum production of entropy** with ONO clarification of the relation with least dissipation of energy.
- **GLANSDORFF and PRIGOGINE's integrale principle** when the conductivity coefficients are not constant

COQUINOT NON-EQUILIBRIUM THERMODYNAMIC THEORY OF DISSIPATIVE BRACKETS

- Coquinot deduced that the **dynamics of out-of-equilibrium thermodynamics** on the phase space can be expressed **with a symmetric bracket** for any two functionals f and g :

$$(f,g)=\frac{1}{\Upsilon}\int_{\Omega}\nabla\left(\frac{\partial f}{\partial\zeta_{a}(y)}\right)L_{\alpha\beta}\nabla\left(\frac{\partial g}{\partial\zeta_{\beta}(y)}\right)d^{3}y$$

- Coquinot has observed that this equation is a **pure geometric object, independent of the basis** $\left\{\zeta_{\alpha}\right\}$, where the functional derivatives can be seen as functional gradients, and both functional gradients are contracted thanks to the pseudometric $\left[L_{\alpha\beta}\right]$ and where **the bracket is symmetric thanks to the Onsager-Casimir relations.**
- As previously, has been demonstrated that:

$$\frac{\partial\sigma(x,t)}{\partial t}=\int_{\Omega}\left[\nabla\left(\frac{\partial\sigma(x,t)}{\partial\zeta_{\alpha}(y,t)}\right)L_{\alpha\beta}(y,t)\nabla\left(\frac{\partial S(t)}{\partial\zeta_{\beta}(y,t)}\right)\right]d^{3}y=\Upsilon(\sigma,S)$$

COQUINOT NON-EQUILIBRIUM THERMODYNAMIC THEORY OF DISSIPATIVE BRACKETS

- Baptiste Coquinot proves a **formal equivalence between the classical out-of-equilibrium thermodynamics and a subclass of metriplectic dynamical systems**, showing that the pseudometric nature of the dissipative bracket, usually an ad hoc hypothesis, is the **exact transcription of the well-known second law of thermodynamics and Onsager's relations** through this equivalence.
- Coquinot's construction shows that the dissipative brackets are completely natural for non-equilibrium thermodynamics, just as Poisson brackets are natural for Hamiltonian dynamics, **deriving a general dissipative bracket, for the first time, from basic thermodynamic first principles**.
- Baptiste Coquinot has considered **the role of entropy, a Casimir invariant, as counterpart to the role of the Hamiltonian in analytical mechanics**, the non-negativity of the pseudometric ensures the entropy growth, as given by the second law of thermodynamics.


L. ONSAGER


B. COQUINOT

Baptiste Coquinot, Philip J. Morrison, A General Metriplectic Framework With Application To Dissipative Extended Magnetohydrodynamics, Journal of Plasma Physics (2020)

THALES
Building a future we can all trust

Transverse Symplectic Foliation Structure


G. REEB

Foliation

Folation Theory Inventors: Ehresmann & Reeb

- André Haefliger passed away 7th March 2023 : **Haefliger, A.: Naissance des feuilletages d'Ehresmann-Reeb à Novikov. Journal 2(5), 99–110 (2016)**
- G. REEB, **Sur certaines propriétés topologiques des variétés feuilletées (Thèse)**, Hermann 1952. supervised by Charles Ehresmann
- " ***Sur une durée de quarante années l'immeuble s'est édifié; des centaines d'ouvriers ont œuvré. L'édifice n'est pas achevé, mais on peut visiter. Oui, visiter est le mot*** " - Georges REEB

Charles Ehresmann

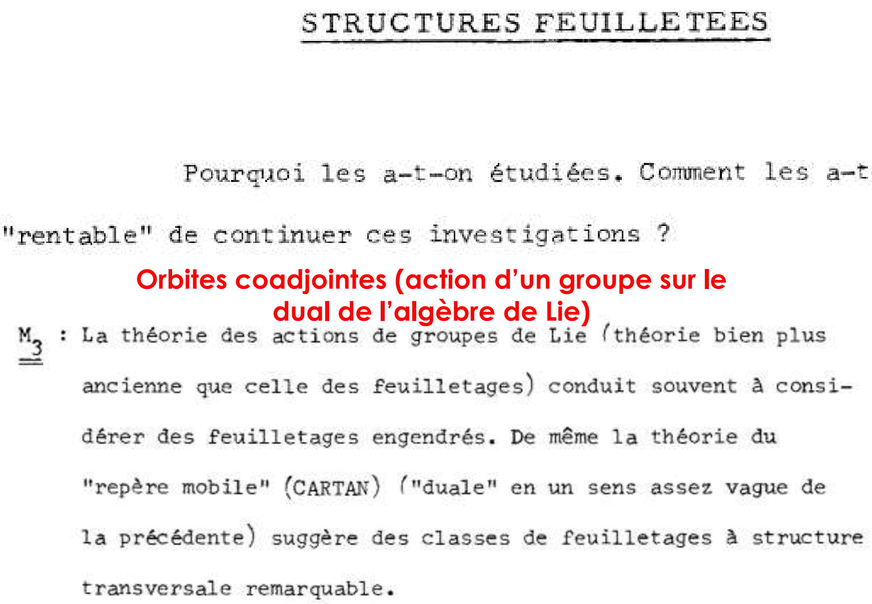
Georges Henri Reeb

Sergei Petrovich Novikov

André Haefliger

Structures feuilletées – Georges Henri REEB
Motivations for Folations Studies

G. Reeb, Structures feuilletées, Differential Topology, Foliations and Gelfand-Fuks cohomology, Rio de Janeiro, 1976, Springer Lecture Notes in Math. 652 (1978), 104-113.



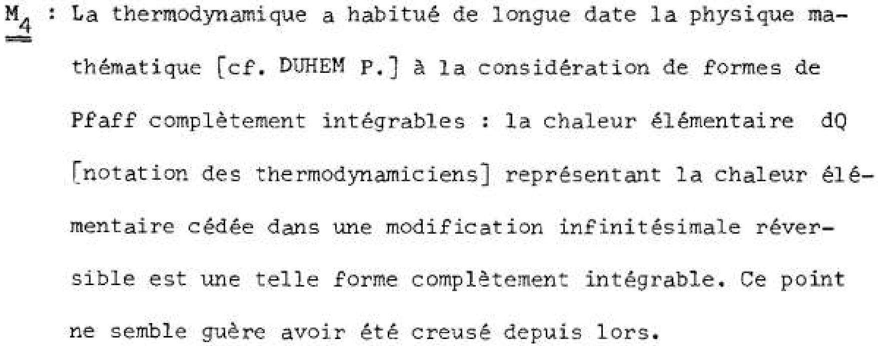
STRUCTURES FEUILLETEES

Pourquoi les a-t-on étudiées. Comment les a-t-on étudiées. Est-il "rentable" de continuer ces investigations ?

Orbites coadjointes (action d'un groupe sur le dual de l'algèbre de Lie)

$\underline{\underline{M_3}}$: La théorie des actions de groupes de Lie (théorie bien plus ancienne que celle des feuilletages) conduit souvent à considérer des feuilletages engendrés. De même la théorie du "repère mobile" (CARTAN) ("duale" en un sens assez vague de la précédente) suggère des classes de feuilletages à structure transversale remarquable.

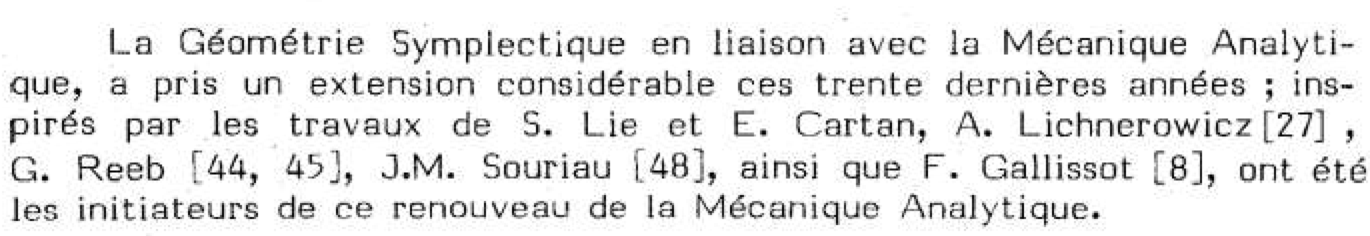
Thermodynamiques et formes de Pfaff



$\underline{\underline{M_4}}$: La thermodynamique a habitué de longue date la physique mathématique [cf. DUHEM P.] à la considération de formes de Pfaff complètement intégrables : la chaleur élémentaire dQ [notation des thermodynamiciens] représentant la chaleur élémentaire cédée dans une modification infinitésimale réversible est une telle forme complètement intégrable. Ce point ne semble guère avoir été creusé depuis lors.

> "*Sur une durée de quarante années l'immeuble s'est édifié; des centaines d'ouvriers ont œuvré. L'édifice n'est pas achevé, mais on peut visiter. Oui, visiter est le mot*" - Georges REEB

Georges REEB contribution to Analytical Mechanics (in « *Œuvres complètes de Charles Ehresmann* »)



La Géométrie Symplectique en liaison avec la Mécanique Analytique, a pris un extension considérable ces trente dernières années ; inspirés par les travaux de S. Lie et E. Cartan, A. Lichnerowicz [27] , G. Reeb [44, 45], J.M. Souriau [48], ainsi que F. Gallissot [8], ont été les initiateurs de ce renouveau de la Mécanique Analytique.

References:

- G. REEB, Propriétés topologiques des trajectoires des systèmes dynamiques, Mém. Acad. Se. Bruxelles 27 (1952).
- G. REEB, Varietés symplectiques, varietés presque-complexes et systèmes dynamiques, Comptes Rendus de l'Académie de Sciences, 235, (1952) 776–778
- G, REEB, Espaces de Finsler et espaces de Cartan, Coll. Int. CNRS Géom. Diff. Strasbourg (1953), 35-40.
- G. REEB, Problèmes relatifs aux variétés presque symplectiques et systèmes dynamiques, Convegno Int. Geom. Diff. Venise-Bologne-Pise (1953), 104-113

Symplectic Leaves
& transverse Metric Leaves

PHYSICAL REVIEW E **91**, 042138 (2015)

Essential equivalence of the general equation for the nonequilibrium reversible-irreversible coupling (GENERIC) and steepest-entropy-ascent models of dissipation for nonequilibrium thermodynamics

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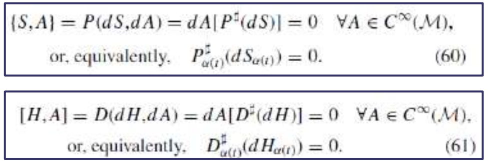
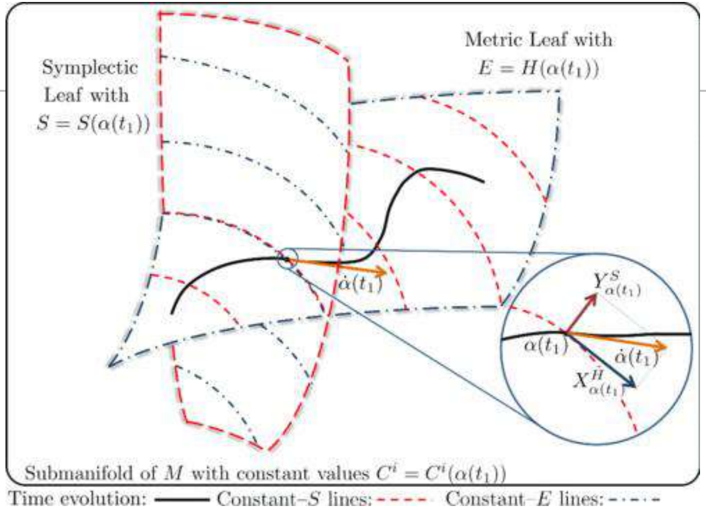
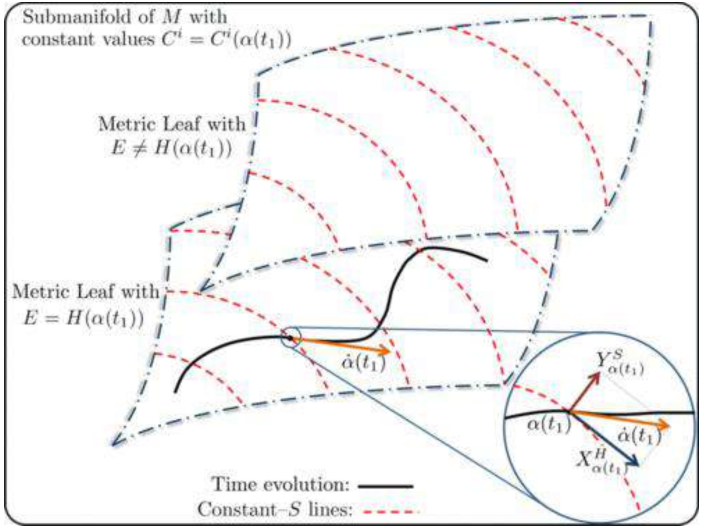
(Received 19 November 2014; published 28 April 2015)



$$\{S,A\} = P(dS,dA) = dA[P^\sharp(dS)] = 0 \quad \forall A \in C^\infty(\mathcal{M}),$$
$$\text{or, equivalently,} \quad P^\sharp_{\alpha(t)}(dS_{\alpha(t)}) = 0. \qquad (60)$$

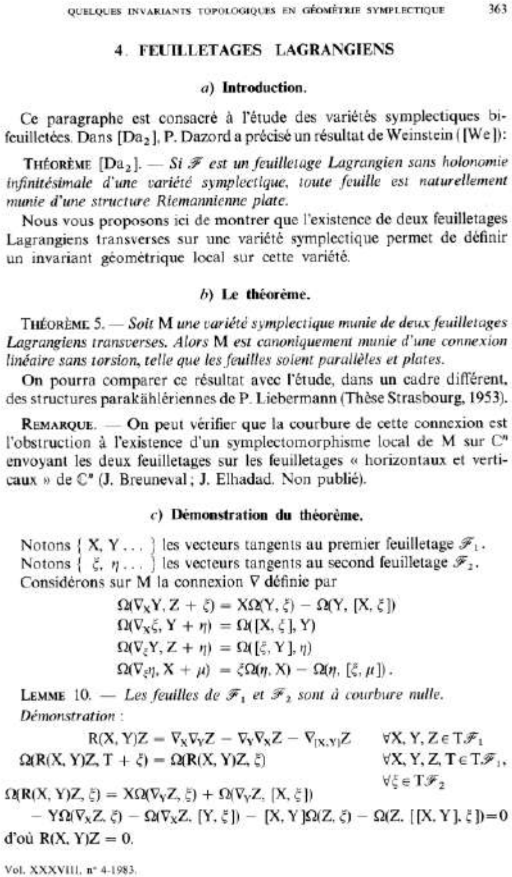
$$[H,A] = D(dH,dA) = dA[D^\sharp(dH)] = 0 \quad \forall A \in C^\infty(\mathcal{M}),$$
$$\text{or, equivalently,} \quad D^\sharp_{\alpha(t)}(dH_{\alpha(t)}) = 0. \qquad (61)$$





- In the context of GENERIC dynamics, the degeneracy condition (60) implies that **symplectic leaves are at constant entropy** (and the other distinguished functionals of the Poisson bracket) while the degeneracy condition (61) implies that **metric leaves are at constant energy** (and the other distinguished functionals of the dissipative bracket).
- Each trajectory is effectively constrained on a single metric leaf.
- The relationship between metric leaves, where GENERIC dynamics (of an overall closed and isolated thermodynamic system) takes place, and symplectic leaves, where purely Hamiltonian dynamics takes place.
- **Metric leaves are surfaces with constant energy, while the symplectic leaves are surfaces with constant entropy** (because Hamiltonian dynamics is reversible).
- **The intersection of symplectic leaves on a metric leaf produces isentropic contours** and the GENERIC non dissipative vector (for an overall closed and isolated thermodynamic system) is always contained in such an intersection.

Jean-Marie MORVAN



QUELQUES INVARIANTS TOPOLOGIQUES EN GÉOMÉTRIE SYMPLECTIQUE 363

4. FEUILLETAGES LAGRANGIENS

a) **Introduction.**

Ce paragraphe est consacré à l'étude des variétés symplectiques bi-feuilletées. Dans [Da$_2$], P. Dazord a précisé un résultat de Weinstein ([We]):

Théorème [Da$_2$]. — *Si $\mathscr{F}$ est un feuilletage Lagrangien sans holonomie infinitésimale d'une variété symplectique, toute feuille est naturellement munie d'une structure Riemannienne plate.*

Nous vous proposons ici de montrer que l'existence de deux feuilletages Lagrangiens transverses sur une variété symplectique permet de définir un invariant géométrique local sur cette variété.

b) **Le théorème.**

Théorème 5. — *Soit* M *une variété symplectique munie de deux feuilletages Lagrangiens transverses. Alors* M *est canoniquement munie d'une connexion linéaire sans torsion, telle que les feuilles soient parallèles et plates.*

On pourra comparer ce résultat avec l'étude, dans un cadre différent, des structures parakählériennes de P. Liebermann (Thèse Strasbourg, 1953).

Remarque. — On peut vérifier que la courbure de cette connexion est l'obstruction à l'existence d'un symplectomorphisme local de M sur $\mathbb{C}^n$ envoyant les deux feuilletages sur les feuilletages « horizontaux et verticaux » de $\mathbb{C}^n$ (J. Breuneval ; J. Elhadad. Non publié).

c) **Démonstration du théorème.**

Notons $\{X, Y \ldots\}$ les vecteurs tangents au premier feuilletage $\mathscr{F}_1$.
Notons $\{\xi, \eta \ldots\}$ les vecteurs tangents au second feuilletage $\mathscr{F}_2$.
Considérons sur M la connexion ∇ définie par

$$\begin{aligned}
\Omega(\nabla_X Y, Z+\xi) &= X\Omega(Y,\xi) - \Omega(Y,[X,\xi])\\
\Omega(\nabla_X \xi, Y+\eta) &= \Omega([X,\xi],Y)\\
\Omega(\nabla_\xi Y, Z+\eta) &= \Omega([\xi,Y],\eta)\\
\Omega(\nabla_\xi \eta, X+\mu) &= \xi\Omega(\eta,X) - \Omega(\eta,[\xi,\mu]).
\end{aligned}$$

Lemme 10. — *Les feuilles de $\mathscr{F}_1$ et $\mathscr{F}_2$ sont à courbure nulle.*

Démonstration :

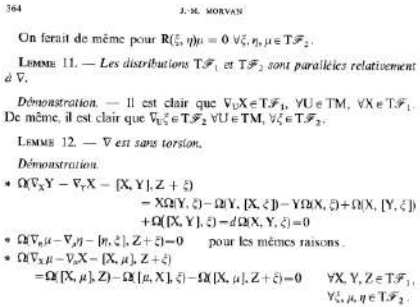
$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \qquad \forall X, Y, Z \in T\mathscr{F}_1$$

$$\Omega(R(X,Y)Z, T+\xi) = \Omega(R(X,Y)Z,\xi) \qquad \forall X, Y, Z, T \in T\mathscr{F}_1,\ \forall \xi \in T\mathscr{F}_2$$

$$\Omega(R(X,Y)Z,\xi) = X\Omega(\nabla_Y Z,\xi) + \Omega(\nabla_Y Z,[X,\xi]) - Y\Omega(\nabla_X Z,\xi) - \Omega(\nabla_X Z,[Y,\xi]) - [X,Y]\Omega(Z,\xi) - \Omega(Z,[[X,Y],\xi]) = 0$$

d'où $R(X,Y)Z = 0$.

Vol. XXXVIII, n° 4-1983.



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On ferait de même pour $R(\xi,\eta)\mu = 0\ \forall \xi, \eta, \mu \in T\mathscr{F}_2$.

Lemme 11. — *Les distributions $T\mathscr{F}_1$ et $T\mathscr{F}_2$ sont parallèles relativement à ∇.*

Démonstration. — Il est clair que $\nabla_U X \in T\mathscr{F}_1$, $\forall U \in TM$, $\forall X \in T\mathscr{F}_1$. De même, il est clair que $\nabla_U \xi \in T\mathscr{F}_2$ $\forall U \in TM$, $\forall \xi \in T\mathscr{F}_2$.

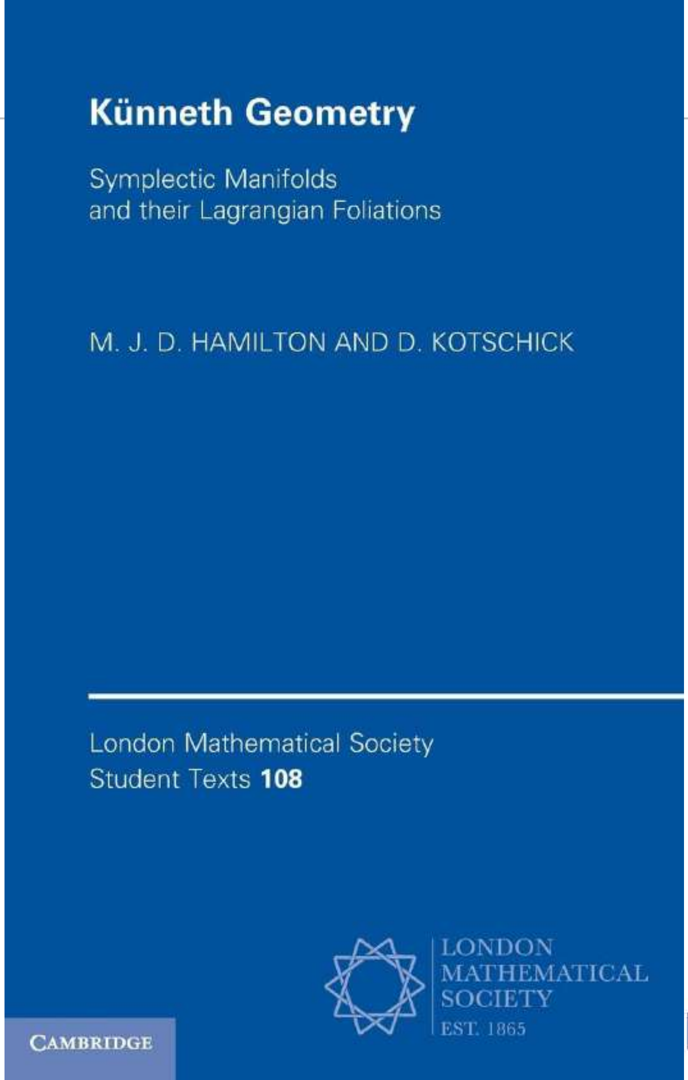
Lemme 12. — *∇ est sans torsion.*

Démonstration.

$$\ast\ \Omega(\nabla_X Y - \nabla_Y X - [X,Y], Z+\xi) = X\Omega(Y,\xi) - \Omega(Y,[X,\xi]) - Y\Omega(X,\xi) + \Omega(X,[Y,\xi]) + \Omega([X,Y],\xi) - d\Omega(X,Y,\xi) = 0$$

$$\ast\ \Omega(\nabla_\eta \mu - \nabla_\mu \eta - [\eta,\xi], Z+\xi) = 0$$ pour les mêmes raisons.

$$\ast\ \Omega(\nabla_X \mu - \nabla_\mu X - [X,\mu], Z+\xi) = \Omega([X,\mu],Z) - \Omega([\mu,X],\xi) - \Omega([X,\mu],Z+\xi) = 0 \qquad \forall X, Y, Z \in T\mathscr{F}_1,\ \forall \xi, \mu, \eta \in T\mathscr{F}_2.$$



Künneth Geometry

Symplectic Manifolds
and their Lagrangian Foliations

M. J. D. HAMILTON AND D. KOTSCHICK

London Mathematical Society
Student Texts **108**

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CAMBRIDGE

Transverse Foliation Structures

- Dazord, P., Molino, P. : Γ-Structures poissonniennes et feuilletages de Libermann, Publications du Département de Mathématiques de Lyon, fascicule 1B, « Séminaire Sud-Rhodanien 1ère partie », chapitre II , p. 69-89 (1988)

Libermann Foliation and Haefliger Γ-Structure

- Sabourin, H. : Quelques aspects de la méthode des orbites en théorie de Lie, HDR spécialité Mathématique, Université de Poitiers, 15 Décembre 2005

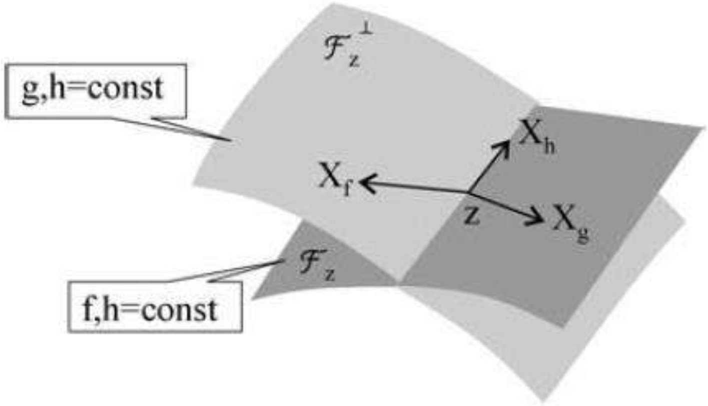
Hervé Sabourin Transverse Poisson Structure

Foliation based on Elie Cartan Moving Frame

- Fedida, E. : Sur la théorie des feuilletages associée au repère mobile : cas des feuilletages de Lie. Lecture Notes in Mathematics, vol 652. Springer (1978)

Polar Foliation, Bifoliation and Bifibration

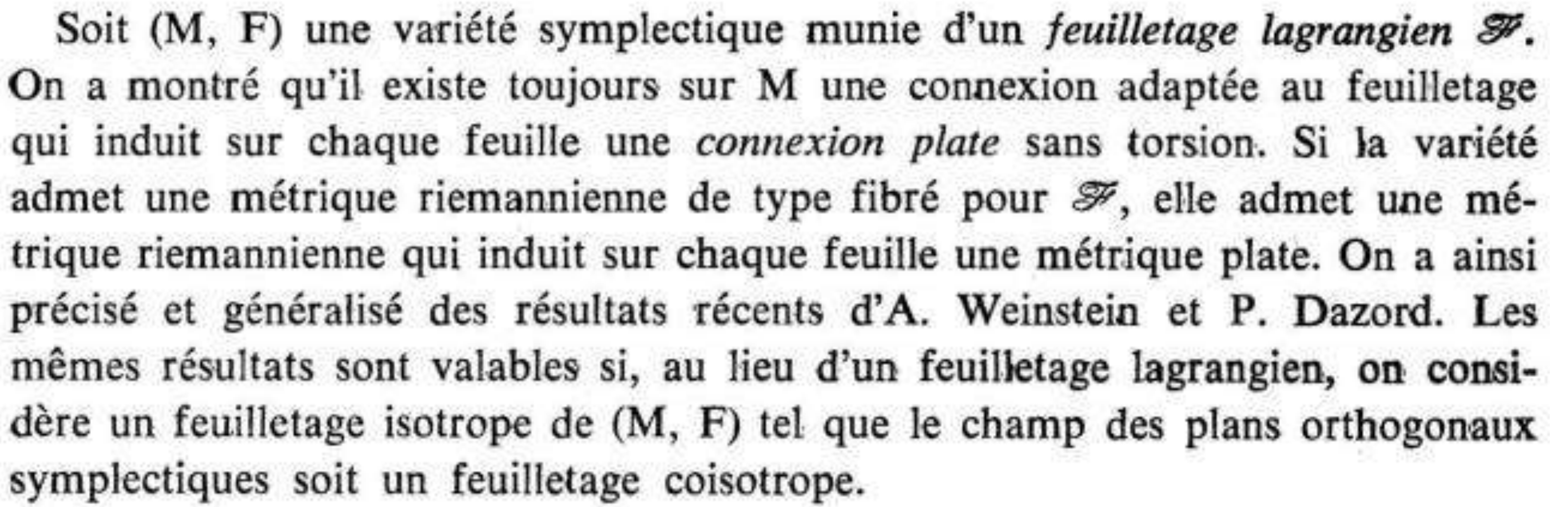
- Arutyunov, G., Elements of Classical and Quantum Integrable Systems, Springer, (2019)



André Lichnerowicz, Collège de France

Cours 1982-1983 – Étude systématique des rapports entre feuilletages et variétés de Poisson | Étude de cohomologies d'algèbres de Lie attachées à une variété de contact

- https://www.college-de-france.fr/sites/default/files/media/document/2023-05/1982-1983_lichnerowicz.pdf



Soit (M, F) une variété symplectique munie d'un *feuilletage lagrangien* $\mathscr{F}$. On a montré qu'il existe toujours sur M une connexion adaptée au feuilletage qui induit sur chaque feuille une *connexion plate* sans torsion. Si la variété admet une métrique riemannienne de type fibré pour $\mathscr{F}$, elle admet une métrique riemannienne qui induit sur chaque feuille une métrique plate. On a ainsi précisé et généralisé des résultats récents d'A. Weinstein et P. Dazord. Les mêmes résultats sont valables si, au lieu d'un feuilletage lagrangien, on considère un feuilletage isotrope de (M, F) tel que le champ des plans orthogonaux symplectiques soit un feuilletage coisotrope.

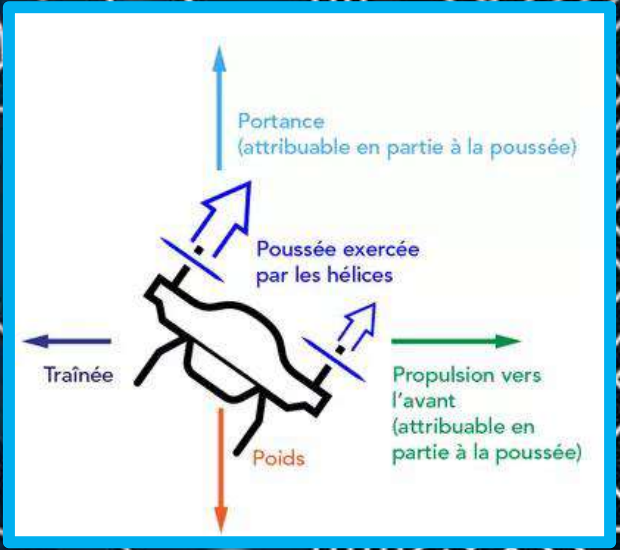
3-web structure and Bi-Lagrangian Manifolds

- This model of web theory was employed by **Gérard Debreu** to define the conditions under which a preference ordering may be represented by a numerical function. Debreu demonstrated that the **condition of additive separability is equivalent to the existence of a trivial 3-web**, which must satisfy <u>the hexagon condition</u> or, equivalently, a **third-order partial differential equation proposed by Paul Samuelson**, also known as the **Saint-Robert equation**.

- This two-form was **first identified by Vilfredo Pareto**. **Paul Samuelson** (Economics Nobel Prize 1970), referred to **Willard Gibbs's thermodynamic mathematical isomorphisms**, studied under his mentor Edwin Wilson and the Harvard Pareto circle. Edwin Wilson had himself previously **studied under Willard Gibbs at Yale**. Samuelson also introduced a **2-web structure**, providing an analytic criterion for determining whether a given 2-web satisfies what is now known as **Samuelson's area condition**, a condition **historically applied by J. C. Maxwell in thermodynamics**. **Tabachnikov** later proved that one may <u>associate the Hess connection, a symplectic, torsion-free connection, with the web, and that Samuelson's condition implies the flatness of this connection.</u>

- Regarding additive separability and integrability, **Thomas Russell observed that the problem of additive separability was addressed by Gérard Debreu**, while **the problem of integrability was treated by Samuelson**. He further noted that each of these problems may be associated with a web whose canonical connection is flat.

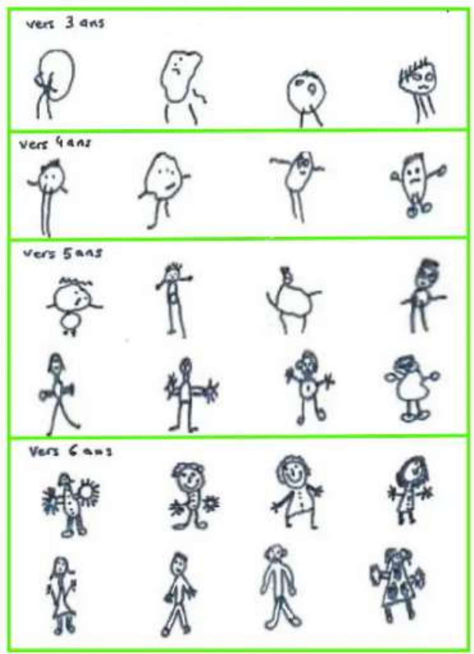
DISSIPATIVE FOR DRONE FLIGHT MECHANICS WITH DRAG & FRICTION FORCES



Diverse Intelligences: Cognitive & Motion Intelligence

Motion Intelligence

Cognitive Intelligence



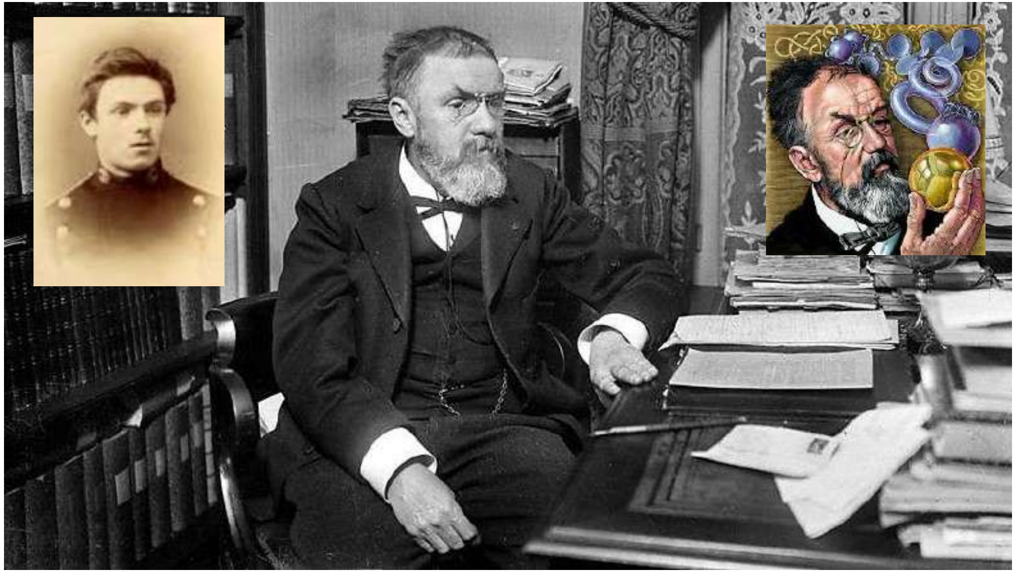
We start to learn space (its invariances) before learning to recognize objects in this space

Diverse Intelligences: Cognitive & Motion Intelligence

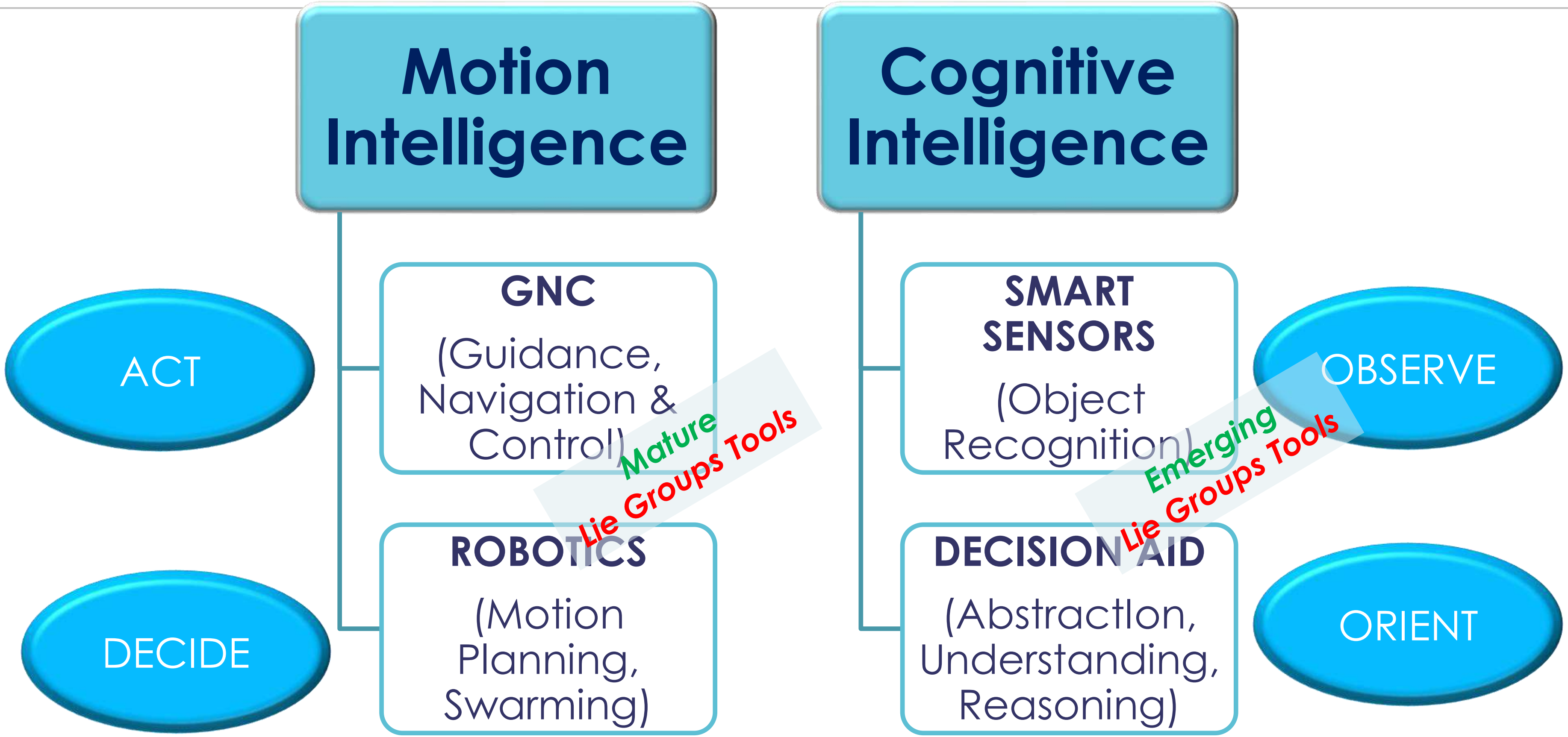
Motion Intelligence

Cognitive Intelligence

John McEnroe: In the Realm of Perfection



Diverse Intelligences: Cognitive & Motion Intelligence



Motion Intelligence

GNC

(Guidance, Navigation & Control)

ROBOTICS

(Motion Planning, Swarming)

ACT

DECIDE

Mature

Lie Groups Tools

Cognitive Intelligence

SMART SENSORS

(Object Recognition)

DECISION AID

(Abstraction, Understanding, Reasoning)

OBSERVE

ORIENT

Emerging

Lie Groups Tools

1 Centrury of Motion Intelligence: Human High Jump

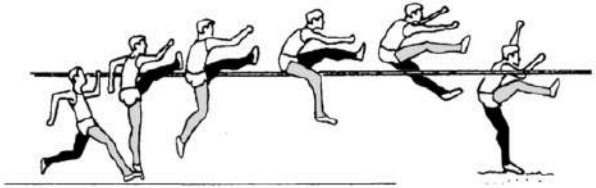
-saut **de face** (du tout début jusque fin 19^{e}) : perfs autour de 1.80m.

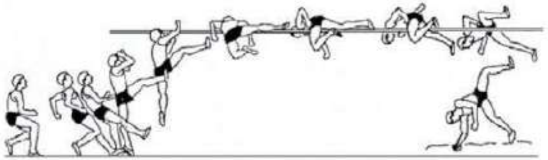
-saut en **ciseau** (période autour de 1900) : perfs autour de 1.90m.

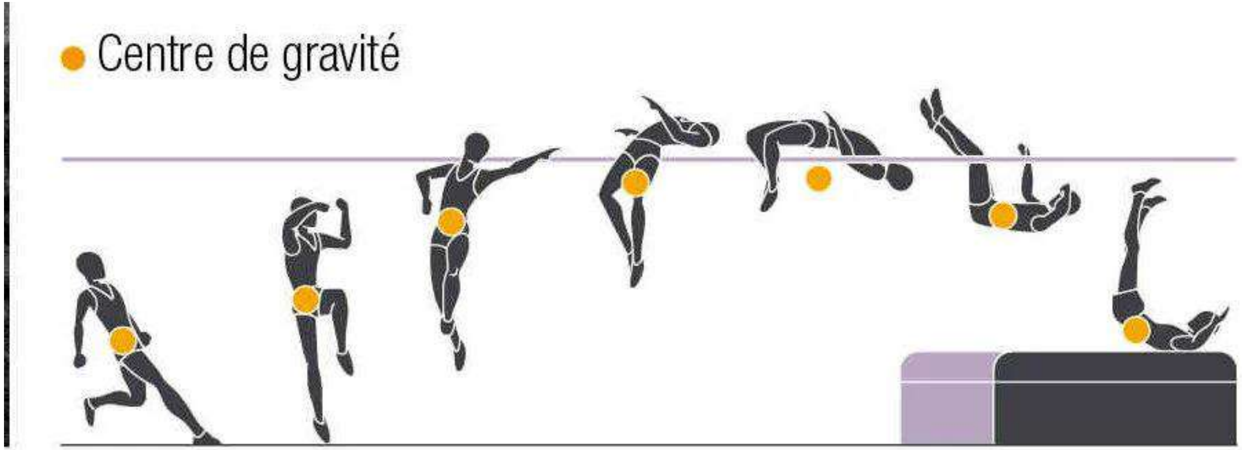
-saut par **retournement intérieur ou extérieur**, **rouleau californien** (début 20^{e} jusque fin des années 30) : 2m franchis en 1912 (Horine) puis progrès réguliers mais assez faibles ensuite.

-**rouleau ventral** (après guerre jusque fin des années 60) : grande évolution jusque 2.28 en 1963 (Brumel).

-**saut dorsal** ou « **Fosbury flop**» inventé en 1968. On passera 2.30 en 1973, 2.40m en 1985, puis 2.45m en 1993 (RM actuel par Sotomayor).







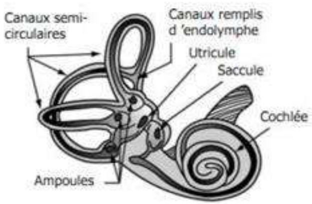

Paul Souriau

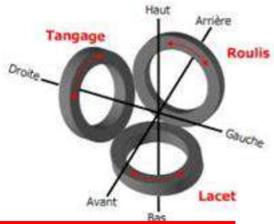
L'esthétique du mouvement

Elibron Classics

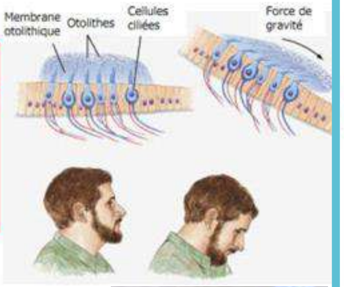
Lie Group in Visio Inertial Brain System

Geolocalization and Navigation :
Visio-Inertial SLAM: Visio-Vestibular Brain System

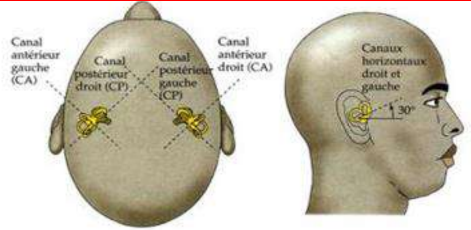


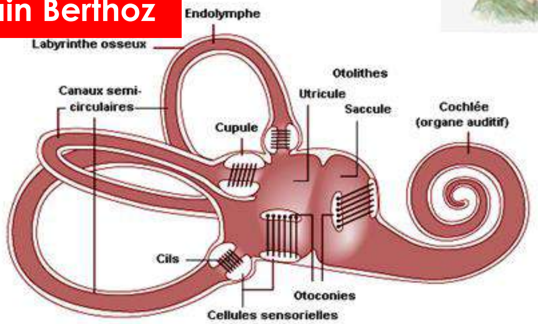


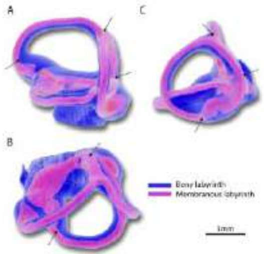
Coding of Homogeneous Galileo Group By Vestibular System and Otolithes



Works of Daniel Bennequin & Alain Berthoz

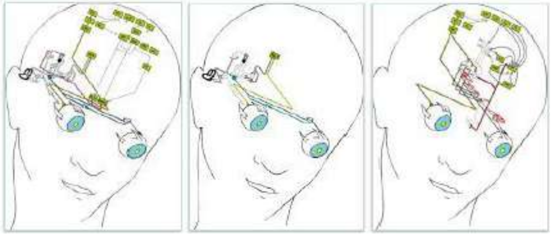






$$\begin{bmatrix} Z' \\ 1 \end{bmatrix} = \begin{bmatrix} \Omega & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Z \\ 1 \end{bmatrix}$$

$$\Omega \in SO(3), t \in R^3$$



https://github.com/HTLife/VINet

VINet: Visual-Inertial Odometry as a Sequence-to-Sequence Learning Problem

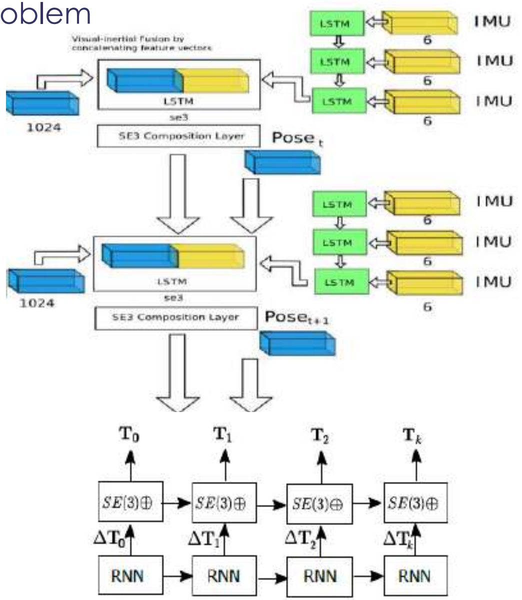
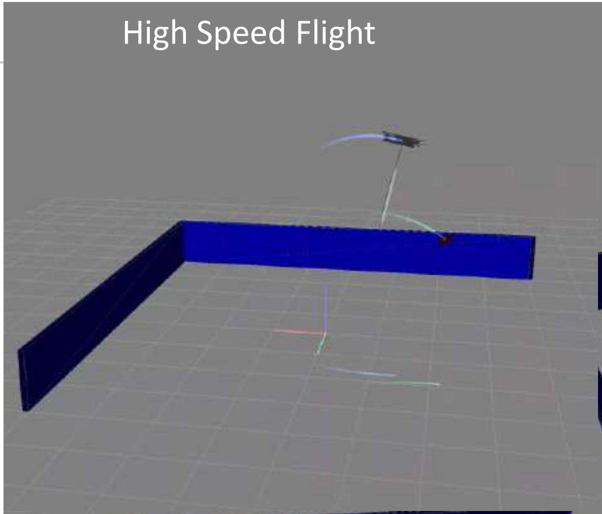
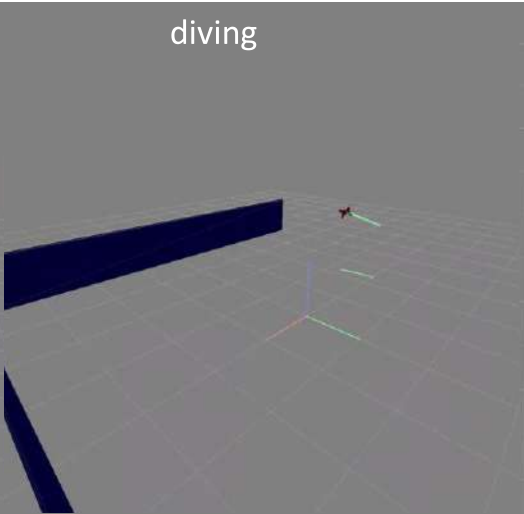
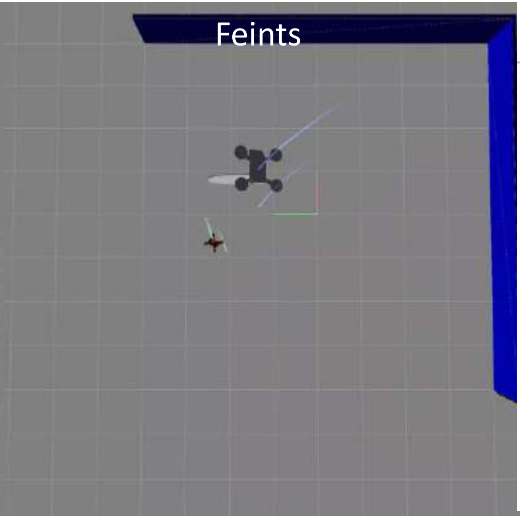
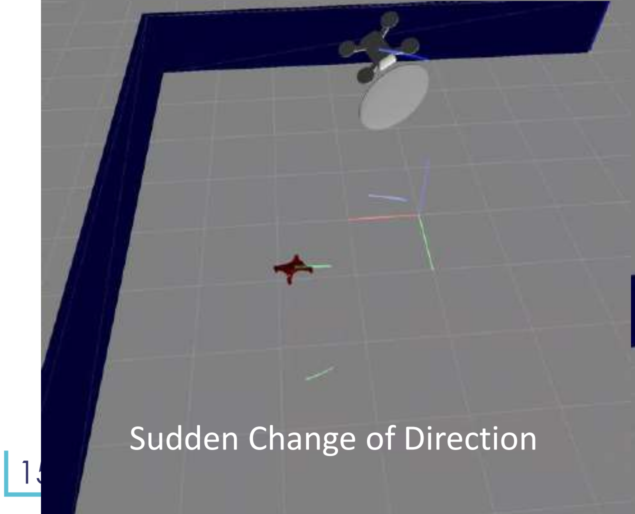


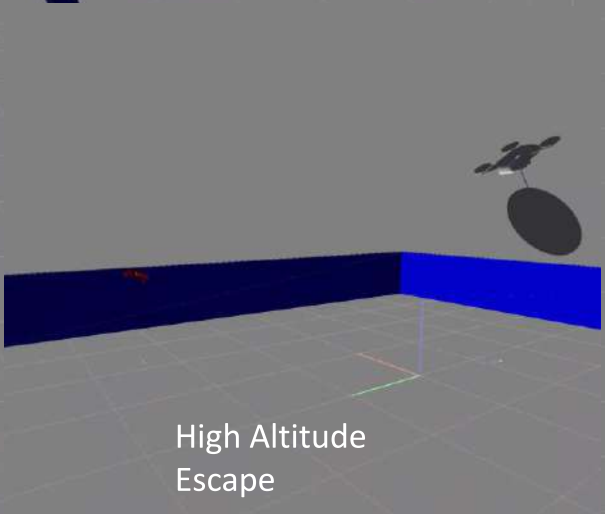
Illustration of the SE(3) composition layer - a parameter-free layer which concatenates transformations between frames on SE(3).

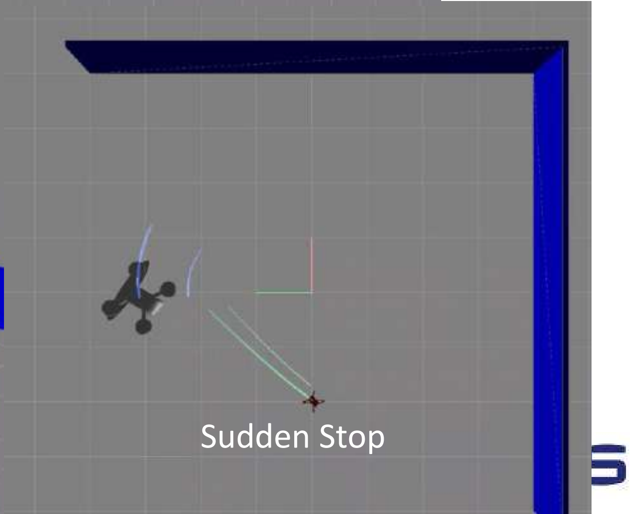

High Speed Flight


diving

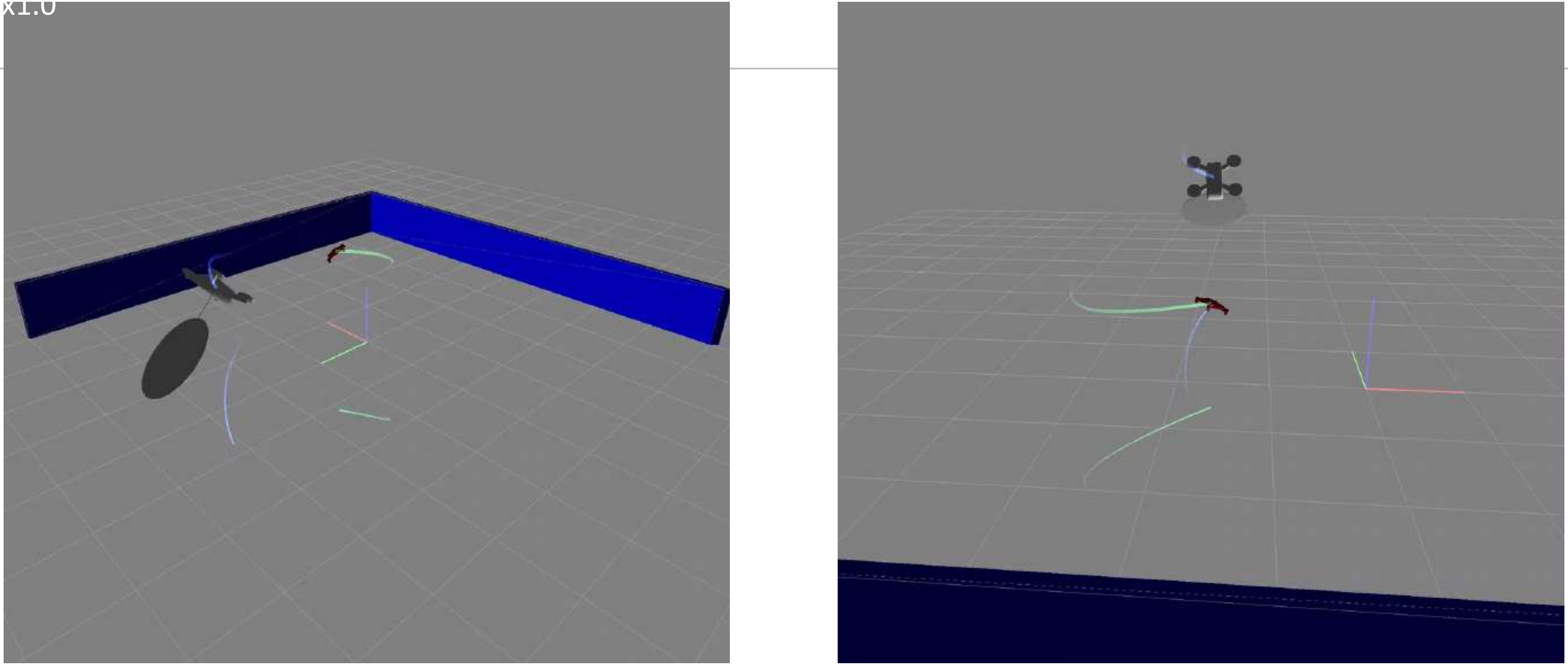

Feints


Sudden Change of Direction

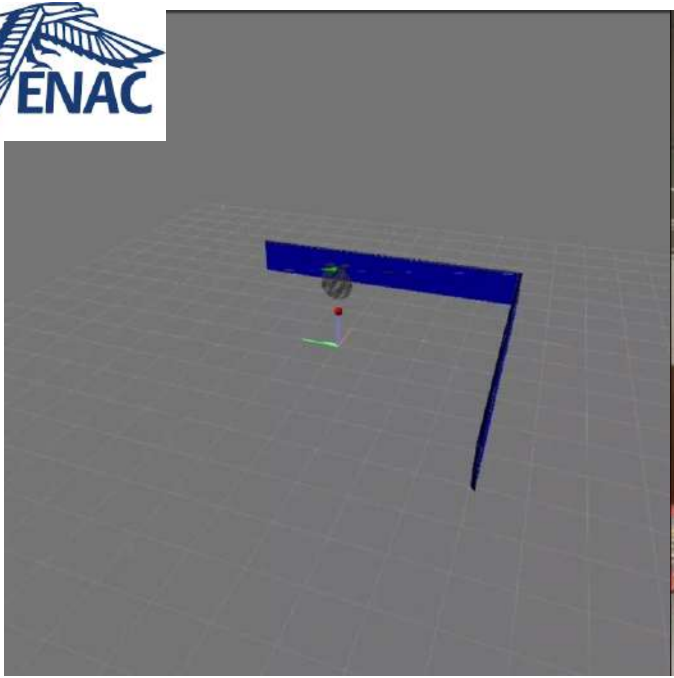

High Altitude Escape


Sudden Stop

S x1.0



The pursuer had learned agile interception maneuvers, sometimes managing to catch the escapee with very high roll and pitch angles.



Real-world flights were simulated indoors with a simulated evasive drone. The pursuing drone attempted to avoid the arena's boundaries and get closer to the evasive drone; it intercepted the simulated target 12

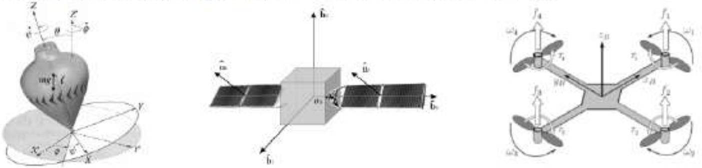
However, we observed a difference in the interception dynamics during the simulation, likely due to a discrepancy between the drone dynamics model used for training the neural network and the actual drone dynamics model.

PHYSICAL AI by PINN (Physics-Informed Neural Network) for Flight

Lie-Poisson Neural Networks (LPNets): Data-based computing of Hamiltonian systems with symmetries.

Extension of PINN Flight Mechanics to the Dissipative Case (Taking Drag into Account)

Systems on Lie groups: rigid body dynamics (drones, satellites), underwater vehicles, fluids and plasma, quantum-classical systems, ...



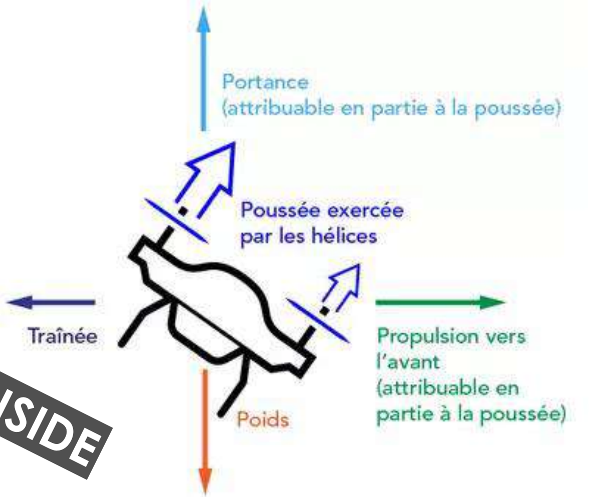
$g \in G$, $G = SO(3)$, $G = SE(3)$, $G = SU(n)$,...

$$\delta \int_0^T L(g,\dot{g})\mathrm{d}t = 0, \quad L = K - U.$$

Left-invariance (e.g., invariance under spatial rotation)

$$L(hg, h\dot{g}) = L(g,\dot{g}), \quad \text{for all } h \in G \quad \Rightarrow \quad L(g,\dot{g}) = \ell(\omega = g^{-1}\dot{g})$$

PHYSICAL AI INSIDE



THALES Leadership on DISSIPATIVE PINN & PHYSICAL AI by TINN & SINN (Thermodynamics & Symplectic-Informed Neural Networks)

Motivation for studying Dissipative PINNs

Goal: embed a physics model in the network architecture to learn physically consistent solutions and dynamics of observed systems

Use cases

Trajectory generation and control

Consistent prediction system dynamics

Parameter identification

Infer mass, inertia, drag, wind force, … from observed data

Anomaly detection

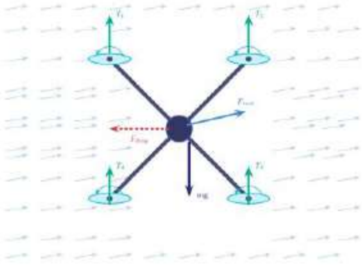
Identify deviations from expected behavior

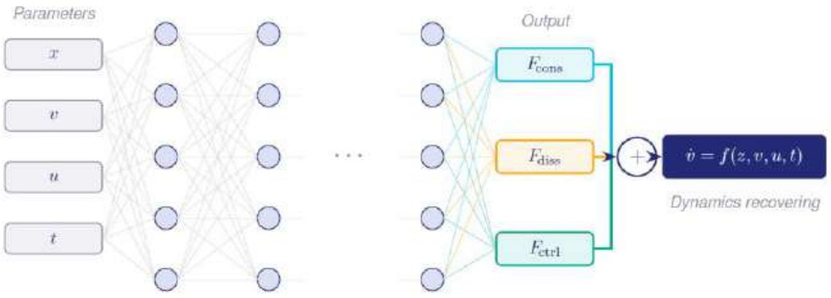
Why dissipative PINNs?

Conservative PINNs: embed the geometry / structure of conservative systems (Hamiltonian, Lagrangian)

Dissipative PINNs: must be informed on the structure / geometry of dissipation — not a black box

Quadcopter dynamics — wind forces, aerodynamic drag, motor losses are not negligible





Equations for the quadcopter

Hamiltonian

$$\begin{aligned}
\dot{\boldsymbol{\Pi}} &= -\frac{\partial h}{\partial \boldsymbol{\Pi}} \times \boldsymbol{\Pi} - \frac{\partial h}{\partial \mathbf{y}} \times \mathbf{y} - \frac{\partial h}{\partial \mathbf{p}} \times \mathbf{p} - \frac{\partial h}{\partial \boldsymbol{\Gamma}} \times \boldsymbol{\Gamma} + \mathbf{T}_{\boldsymbol{\Pi}}(\omega) - \mathbf{D}_{\boldsymbol{\Pi}}(\boldsymbol{\Pi}, \mathbf{p}),\\
\dot{\mathbf{y}} &= -\frac{\partial h}{\partial \boldsymbol{\Pi}} \times \mathbf{y} + \frac{\partial h}{\partial \mathbf{p}},\\
\dot{\mathbf{p}} &= -\frac{\partial h}{\partial \boldsymbol{\Pi}} \times \mathbf{p} - \frac{\partial h}{\partial \mathbf{y}} + \mathbf{T}_{\mathbf{p}}(\omega) - \mathbf{D}_{\mathbf{p}}(\boldsymbol{\Pi}, \mathbf{p}),\\
\dot{\boldsymbol{\Gamma}} &= -\frac{\partial h}{\partial \boldsymbol{\Pi}} \times \boldsymbol{\Gamma}.
\end{aligned}$$

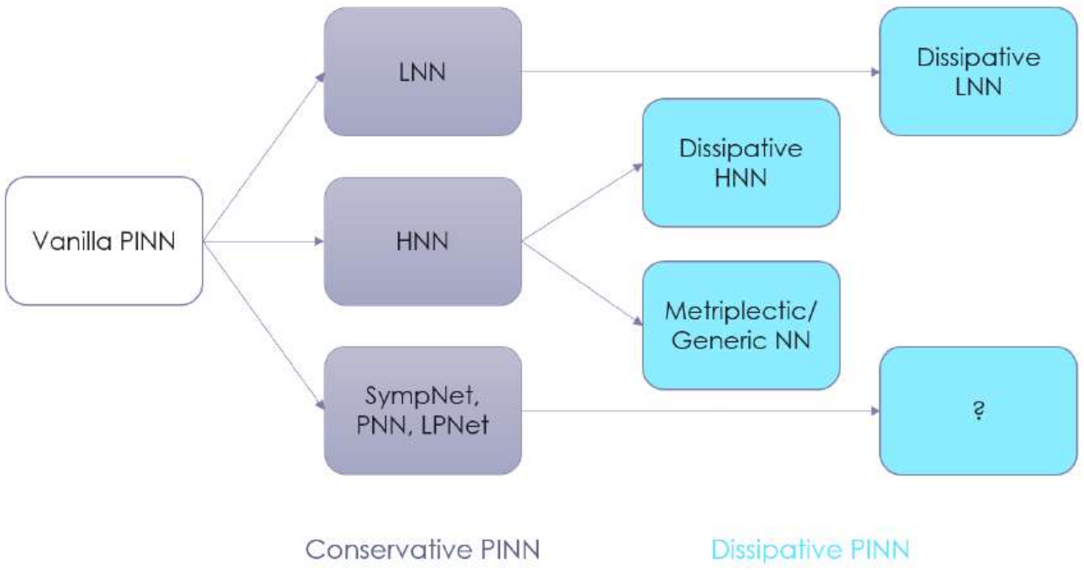
Lagrangian

$$\begin{aligned}
\frac{d}{dt}\frac{\partial \ell}{\partial \boldsymbol{\Omega}} &= -\boldsymbol{\Omega} \times \frac{\partial \ell}{\partial \boldsymbol{\Omega}} + \frac{\partial \ell}{\partial \mathbf{y}} \times \mathbf{y} - \mathbf{w} \times \frac{\partial \ell}{\partial \mathbf{w}} + \frac{\partial \ell}{\partial \boldsymbol{\Gamma}} \times \boldsymbol{\Gamma} + \mathbf{T}_{\boldsymbol{\Omega}}(\omega) - \mathbf{D}_{\boldsymbol{\Omega}}(\boldsymbol{\Omega}, \mathbf{w}),\\
\dot{\mathbf{y}} &= -\boldsymbol{\Omega} \times \mathbf{y} + \mathbf{w},\\
\frac{d}{dt}\frac{\partial \ell}{\partial \mathbf{w}} &= -\boldsymbol{\Omega} \times \frac{\partial \ell}{\partial \mathbf{w}} + \frac{\partial \ell}{\partial \mathbf{y}} + \mathbf{T}_{\mathbf{w}}(\omega) - \mathbf{D}_{\mathbf{w}}(\boldsymbol{\Omega}, \mathbf{w}),\\
\dot{\boldsymbol{\Gamma}} &= -\boldsymbol{\Omega} \times \boldsymbol{\Gamma}.
\end{aligned}$$

Metriplectic ?

In progress, similar to the body in rotation

Families of Dissipative PINNs



Dissipation structure

Black box, potential, ad-hoc, positive semi-definite, metriplectic

Physics constraints enforcing

Soft (loss function) vs Hard constraining (structural)

Integrator

Symplectic, Variational, usual (RK, …), None

All these points strongly influence learning quality. Frameworks must be adapted case by case

Dissipative extension of Hamiltonian Dynamics

Three ways to add dissipation to a Hamiltonian system:

External force

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i} + F_i^{\mathrm{diss}}(q, p)$$

Most general. No structural constraint on F.

Symmetric dissipation

$$\dot{z} = X_H(z) - D(z)\nabla H(z)$$

D symmetric positive semi-definite. Ensures the decrease of energy.

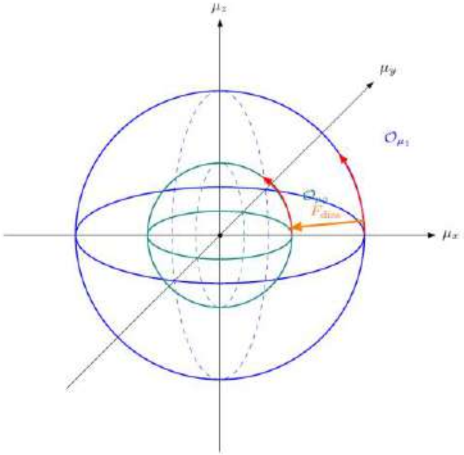
Rayleigh dissipation

$$\dot{q}^i = \frac{\partial H}{\partial p_i} + \frac{\partial R}{\partial q_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i} - \frac{\partial \mathcal{R}}{\partial p^i}$$

Decomposition of the vector field into an irrotational part and a conservative part.

Effect on coadjoint orbits

Dissipation drives drift between orbits. Casimir invariants are not preserved in general.



Metriplectic / Generic formulation

Metriplectic bracket (2-Bracket)

— It modifies the Poisson bracket into a metriplectic bracket (metric + symplectic)
— The flow is generated by a single potential E+S

Energy E and Entropy S

— Energy E is conserved, in the kernel of the metric bracket
— Entropy S is monotonic, Casimir of the Poisson bracket
— Both thermodynamic laws satisfied by construction

Dissipation on SO(3)

— The easiest model is an algebraic dissipation: the kinetic energy is preserved.

— In order to have a real dissipation, one has to add an external entropy variable, counting the energy exchange between the system and its environment.

$$\frac{dF}{dt} = \{F, E\} + (F, S) \qquad \dot{z} = J(z)\nabla E(z) + G(z)\nabla S(z)$$

$$J(z)^T = -J(z), \qquad G(z)^T = G(z), \qquad G(z) \geq 0$$

$$J(z)\nabla S(z) = 0 \qquad G(z)\nabla E(z) = 0 \qquad \{F, S\} = 0 \qquad \forall F$$

$$\frac{dH}{dt} = \{H, H\} + (H, S) = 0$$

$$\frac{dS}{dt} = \{S, H\} + (S, S) \geq 0$$

$$\frac{dL}{dt} = L \times \sigma^{-1} L + 2W'(L^2)\|\sigma^{-1}L\|^2 \pi_{(\sigma^{-1}L)^\perp}(L)$$

$$\frac{dL}{dt} = L \times \sigma^{-1} L - DL$$

$$z = (L, s)$$

$$g = \begin{pmatrix} G_0 & -DL \\ (-DL)^T & \frac{1}{T}(DL)^T(\frac{dH}{dL}) \end{pmatrix}$$

THALES

Metriplectic / Generic Neural Networks

MNN / GFINN principles

— Learn E and S

— Learn Poisson bracket {,}θ (antisymmetric) + metric (,)θ (symmetric positive)

— Enforce (E,.)=0 and {S,.}=0 in the loss (soft) or by a good parametrization (good projections to assure the orthogonality)

Training strategies

— Trajectory fitting: $\mathcal{L}(\theta) = \left\|\mathrm{ODESolver}(f_\theta, z_0, t) - z^{\mathbf{data}}(t)\right\|^2$

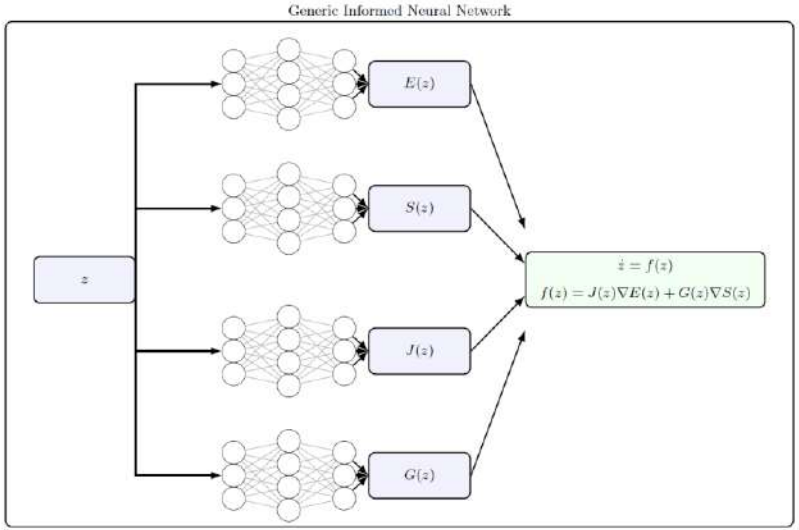
— Velocity fitting: $\mathcal{L}(\theta) = \sum_k \|f_\theta(z_k, u_k) - \dot{z}_k\|^2$

Properties and variants

— Trajectory reconstruction: use a GENERIC integrator, or a simple integrator

— If entropy is not accessible, then one can consider it as a latent variable and project the predictions on the observable variables (DiLar-NN)

MNN architecture



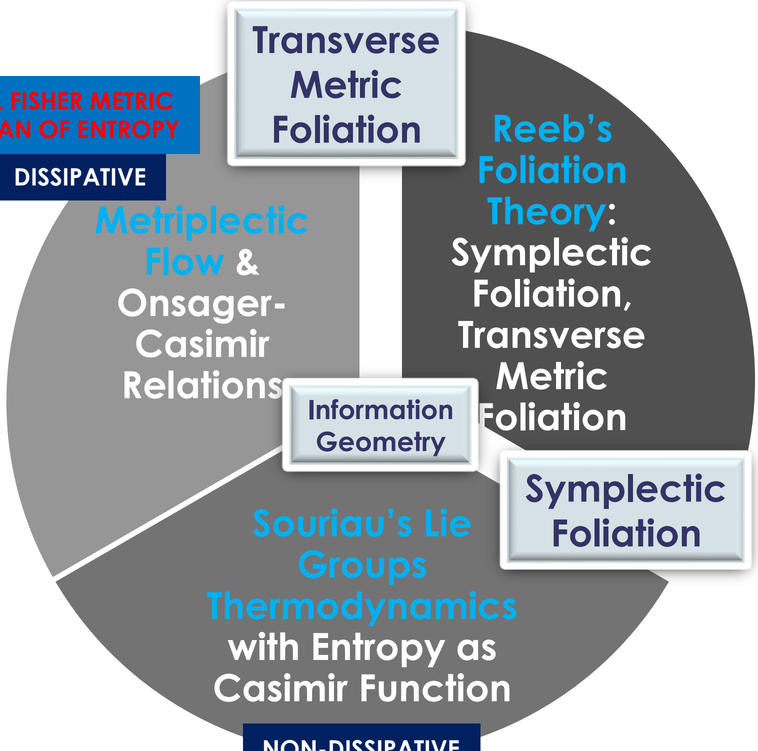
Conclusion

2nd Principle by Henri Poincaré (Bon sens paysan) fable of the grain of oats and the pile of wheat

With his "peasant common sense", **Henri Poincaré described the 2nd principle** in the form of a fable of the grain of oats and the pile of wheat, in the last sentence of his course:

[…] Suppose we want to place a grain of oats in the middle of a pile of wheat; it will be easy; suppose that we then want to find it there and remove it; we will not be able to achieve this. All irreversible phenomena, according to certain physicists, are built on this model [Supposons que nous voulions placer un grain d'avoine au milieu d'un tas de blé ; cela sera facile; supposons que nous voulions ensuite l'y retrouver et l'en retirer; nous ne pourrons y parvenir. Tous les phénomènes irréversibles, d'après certains physiciens, seraient construits sur ce modèle]. (Poincaré 1908)

Symplectic Foliation Transverse Structure of Dissipative Thermodynamics



DUAL FISHER METRIC
HESSIAN OF ENTROPY

DISSIPATIVE

Transverse Metric Foliation

Metriplectic Flow & Onsager-Casimir Relations

Reeb's Foliation Theory: Symplectic Foliation, Transverse Metric Foliation

Information Geometry

Symplectic Foliation

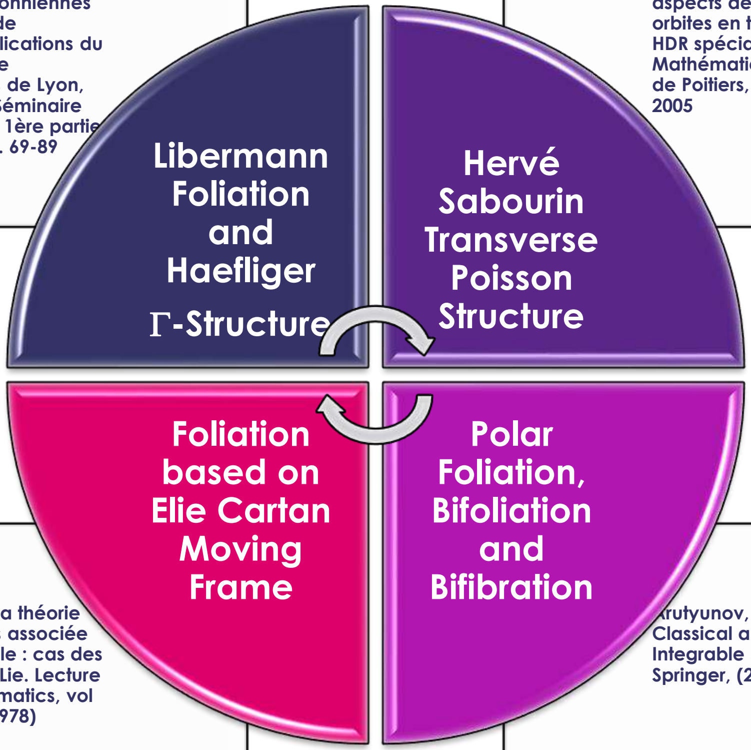
Souriau's Lie Groups Thermodynamics with Entropy as Casimir Function

NON-DISSIPATIVE

FISHER METRIC
HESSIAN OF MASSIEU
KKS 2-FORM

- Dazord, P., Molino, P. : Γ-Structures poissonniennes et feuilletages de Libermann, Publications du Département de Mathématiques de Lyon, fascicule 1B, « Séminaire Sud-Rhodanien 1ère partie », chapitre II , p. 69-89 (1988)

- Sabourin, H. : Quelques aspects de la méthode des orbites en théorie de Lie, HDR spécialité Mathématique, Université de Poitiers, 15 Décembre 2005



Libermann Foliation and Haefliger Γ-Structure

Hervé Sabourin Transverse Poisson Structure

Foliation based on Elie Cartan Moving Frame

Polar Foliation, Bifoliation and Bifibration

- Fedida, E. : Sur la théorie des feuilletages associée au repère mobile : cas des feuilletages de Lie. Lecture Notes in Mathematics, vol 652. Springer (1978)

- rutyunov, G., Elements of Classical and Quantum Integrable Systems, Springer, (2019)

DISSIPATIVE MACHINE LEARNING: BORGES' LEARNING

What is the role of « Dissipation » in Machine Learning ?

Classical Machine Learning

$$\theta_{k+1} = \theta_k - \eta \nabla L(\theta_k) \qquad \dot{\theta} = -\nabla L(\theta) \qquad \frac{dL}{dt} = -\|\nabla L\|^2 \leq 0$$

Dissipation (as friction) in Machine Learning

$$\ddot{\theta} + \gamma\dot{\theta} + \nabla L(\theta) = 0$$

- ∇L = conservative force,
- $\gamma\dot{\theta}$ = friction/dissipation.

$$E = \frac{1}{2}\|\dot{\theta}\|^2 + L(\theta) \qquad \frac{dE}{dt} = -\gamma\|\dot{\theta}\|^2$$

Langevin Dynamics

$$d\theta_t = -\nabla L(\theta_t)dt + \sqrt{2T}\,dW_t$$

- dW_t = Brownian noise,
- T = temperature.

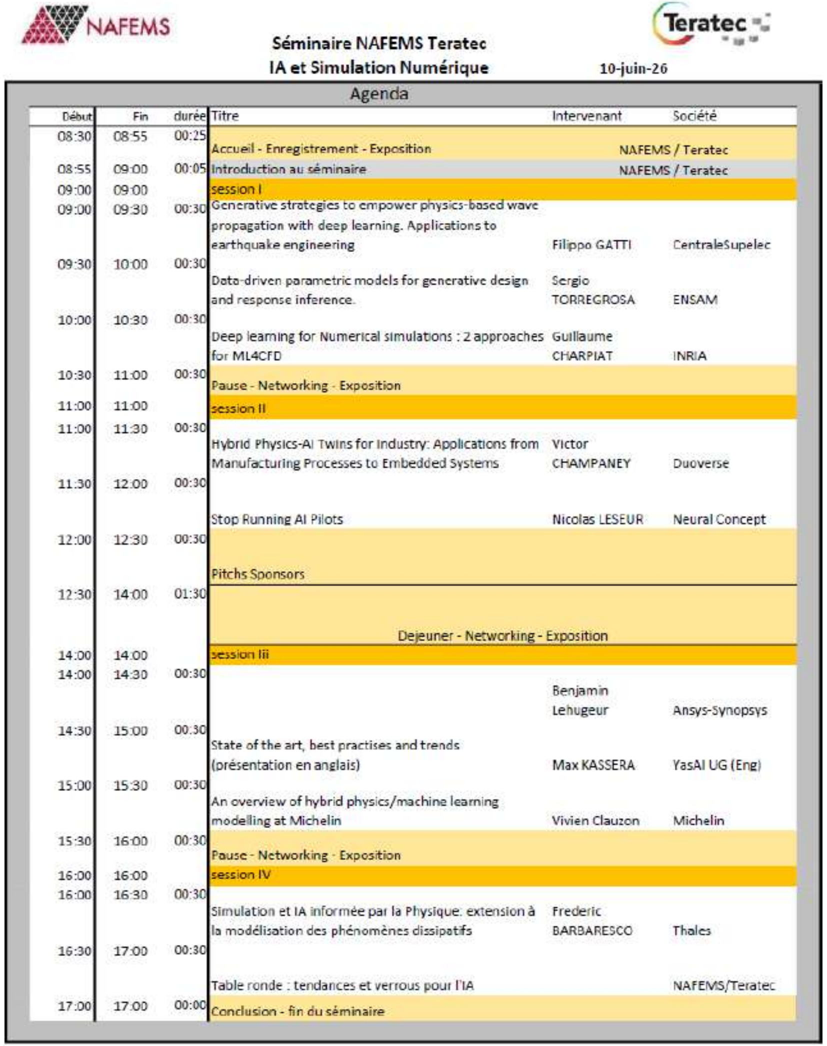
"Forgetting and memory are equally inventive."
[L'oubli et la mémoire sont également inventifs]
Jorge Luis Borges

IAS Frontiers Conference on Geometry, Dynamics, and Learning (GDL2026)

https://www.ntu.edu.sg/spms/news-events/events/detail/2026/09/28/default-calendar/ias-frontiers-conference-on-geometry--dynamics--and-learning

NAFEMS/TERATEC Seminar

- > CENTRALESUPELEC
- > ENSAM
- > INRIA
- > DUOVERSE (spin-off DesCartes CNRS@CREATE)
- > NEURALCONCEPT
- > ANSYS-SYNOPSYS
- > YasAI UG
- > MICHELIN
- > THALES



Séminaire NAFEMS Teratec
IA et Simulation Numérique

10-juin-26

Agenda

Début	Fin	durée	Titre	Intervenant	Société
08:30	08:55	00:25	Accueil - Enregistrement - Exposition		NAFEMS / Teratec
08:55	09:00	00:05	Introduction au séminaire		NAFEMS / Teratec
09:00	09:00		session I		
09:00	09:30	00:30	Generative strategies to empower physics-based wave propagation with deep learning. Applications to earthquake engineering	Filippo GATTI	CentraleSupelec
09:30	10:00	00:30	Data-driven parametric models for generative design and response inference.	Sergio TORREGROSA	ENSAM
10:00	10:30	00:30	Deep learning for Numerical simulations : 2 approaches for ML4CFD	Guillaume CHARPIAT	INRIA
10:30	11:00	00:30	Pause - Networking - Exposition		
11:00	11:00		session II		
11:00	11:30	00:30	Hybrid Physics-AI Twins for Industry: Applications from Manufacturing Processes to Embedded Systems	Victor CHAMPANEY	Duoverse
11:30	12:00	00:30	Stop Running AI Pilots	Nicolas LESEUR	Neural Concept
12:00	12:30	00:30	Pitchs Sponsors		
12:30	14:00	01:30	Dejeuner - Networking - Exposition		
14:00	14:00		session Iii		
14:00	14:30	00:30		Benjamin Lehugeur	Ansys-Synopsys
14:30	15:00	00:30	State of the art, best practises and trends (présentation en anglais)	Max KASSERA	YasAI UG (Eng)
15:00	15:30	00:30	An overview of hybrid physics/machine learning modelling at Michelin	Vivien Clauzon	Michelin
15:30	16:00	00:30	Pause - Networking - Exposition		
16:00	16:00		session IV		
16:00	16:30	00:30	Simulation et IA informée par la Physique: extension à la modélisation des phénomènes dissipatifs	Frederic BARBARESCO	Thales
16:30	17:00	00:30	Table ronde : tendances et verrous pour l'IA		NAFEMS/Teratec
17:00	17:00	00:00	Conclusion - fin du séminaire		

THALES
Building a future we can all trust

Contact

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KTD PCC SENSING SEGMENT LEADER

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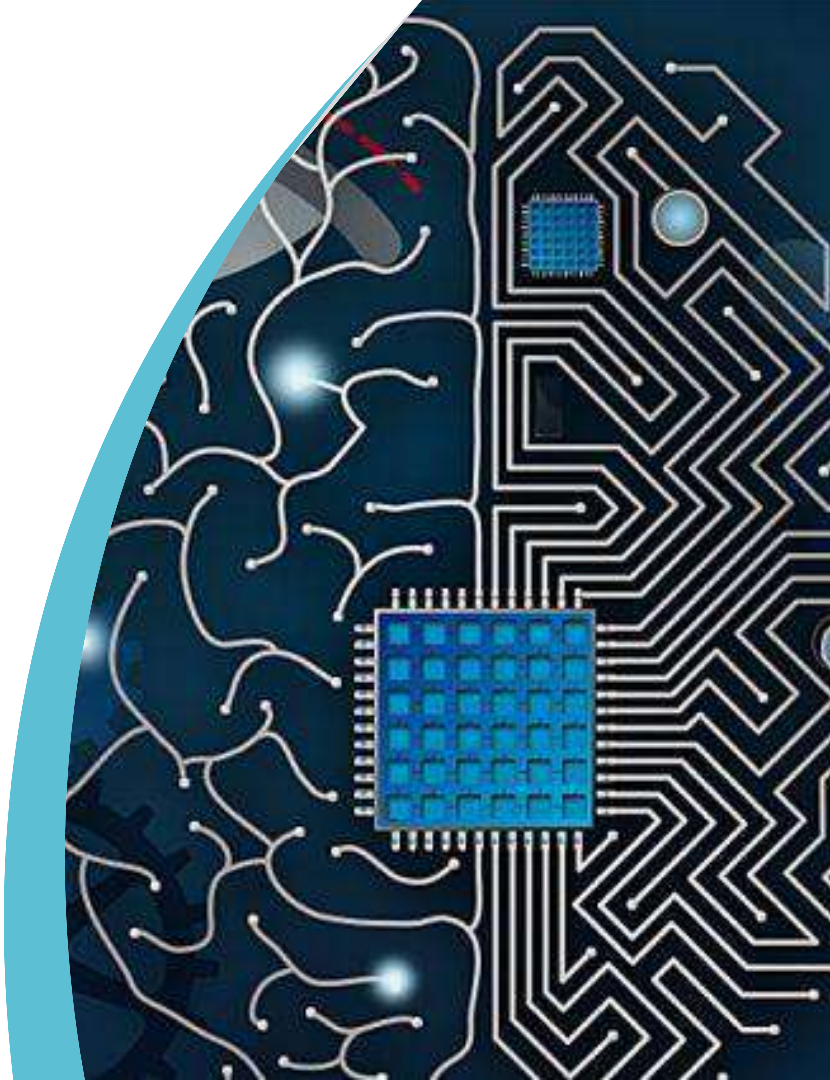
✉ frederic.barbaresco@thalesgroup.com

References



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5. Barbaresco, F., Symplectic Foliation Transverse Structure and Libermann Foliation of Heat Theory and Information Geometry. In: Nielsen, F., Barbaresco, F. (eds) Geometric Science of Information. GSI 2023. Lecture Notes in Computer Science, vol 14072. Springer, Cham. https://doi.org/10.1007/978-3-031-38299-4_17
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Structure de feuilletage symplectique transverse pour la thermodynamique dissipative, fondée sur le modèle géométrique de mécanique statistique de Souriau

Résumé

- Les réseaux de neurones informés par la Physique (PINN « Physics-Informed Neural Network » et HNN « Hamiltonian Neural Network ») sont une alternative pour une IA plus frugale en données et en consommation énergétique en hybridant les modèles analytiques de la physique avec les techniques d'apprentissage. Nous présenterons des outils géométriques venant de la théorie des représentations des groupes de Lie (travaux de Alexandre Kirillov, Bertram Kostant et Jean-Marie Souriau), de la cohomologie des algèbres de Lie (travaux de Jean-Louis Koszul), des structures de feuilletage symplectique (travaux de Charles Ehresmann, Georges Reeb et Paulette Libermann) et de l'équation métriplectique (travaux de Baptiste Coquinot). La géométrie symplectique est la « grammaire » commune des systèmes dynamiques de la physique classique ou quantique. Nous commencerons par présenter des réseaux de neurones équivariants qui utilisent des convolutions sur les groupes de Lie non-compacts pour des applications radars ou caméra hémisphériques. Après avoir introduit le modèle symplectique de la mécanique statistique, appelée « Thermodynamique des groupe de Lie », nous l'interprétons comme une extension de la géométrie de l'information aux groupes de Lie qui donne une définition purement géométrique à l'Entropie. L'Entropie apparait ainsi comme une fonction de Casimir invariante en représentation coadjointe, sur les feuilletages symplectiques générés par les orbites coadjointes via l'application moment (l'application moment étant une géométrisation du théorème d'Emmy Noether). Sur la base des travaux de Paulette Libermann sur les feuilletages Lagrangiens transverses à connexion canonique plate, autour de ce qu'Élie Cartan appelait le « problème d'équivalence » pour s'affranchir du choix de tout système de coordonnées, nous étendons le modèle de Jean-Marie Souriau de la « Thermodynamique des groupes de Lie » au cas dissipatif via un feuilletage Riemannien, s'interprétant comme les courbes de niveaux de l'énergie, transverse au feuilletage symplectique précédent correspondant aux courbes de niveau de l'entropie ; donnant ainsi une interprétation géométrique au 2nd principe de la thermodynamique de Sadi Carnot. Nous illustrons ce modèle symplectique de la dissipation pour les réseaux de neurones informés par la thermodynamique (TINN « Thermodynamics Informed Neural Networks »). Nous montrerons comment ces modèle dissipatifs peuvent être utilisés par des réseaux de neurones informés par la physique pour le cas de la mécanique de vol des drones en modélisant le phénomène dissipatif de la trainée. Nous conclurons par une question ouverte « quel rôle joue la dissipation dans les processus d'apprentissage sachant que toutes les techniques existantes d'intelligence artificielle sont basées sur des gradients d'apprentissage non-dissipatifs ? »..

Structure de feuilletage symplectique transverse pour la thermodynamique dissipative, fondée sur le modèle géométrique de mécanique statistique de Souriau

Principales références bibliographiques

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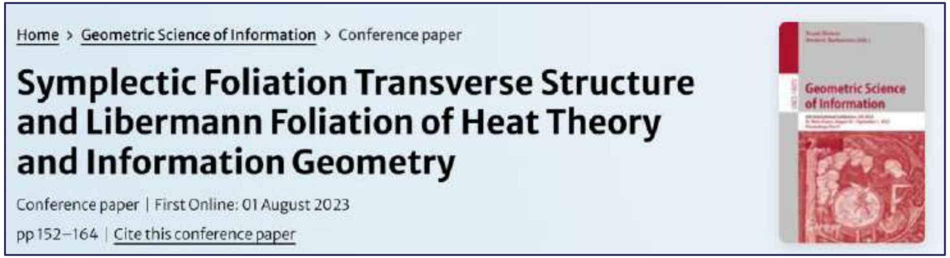

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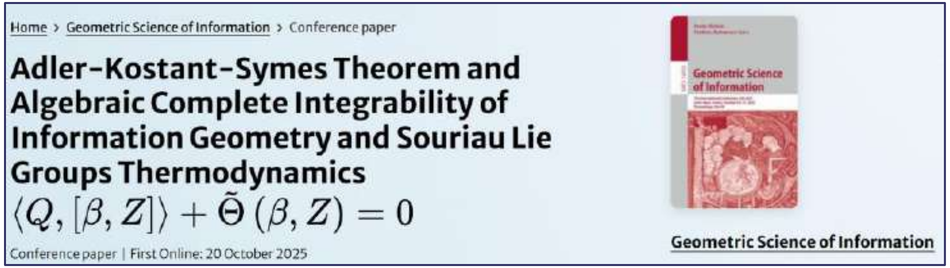
Conference paper | First Online: 01 August 2023

pp 152–164 | Cite this conference paper



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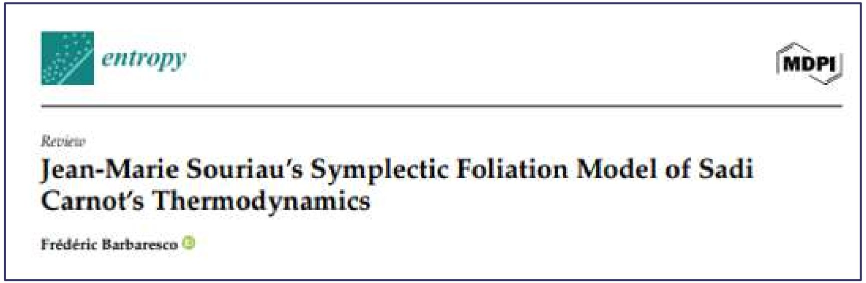

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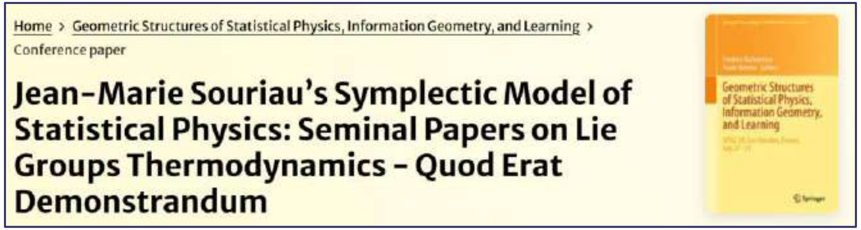
Review

Jean-Marie Souriau's Symplectic Foliation Model of Sadi Carnot's Thermodynamics

Frédéric Barbaresco



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Symplectic Foliation Structures of Non-Equilibrium Thermodynamics as Dissipation Model: Application to Metriplectic Nonlinear Lindblad Quantum Master Equation

by Frédéric Barbaresco

THALES Land & Air Systems, 19/21 Avenue Morane Saulnier, 78140 Vélizy-Villacoublay, France

Entropy **2022**, *24*(11), 1626; https://doi.org/10.3390/e24111626

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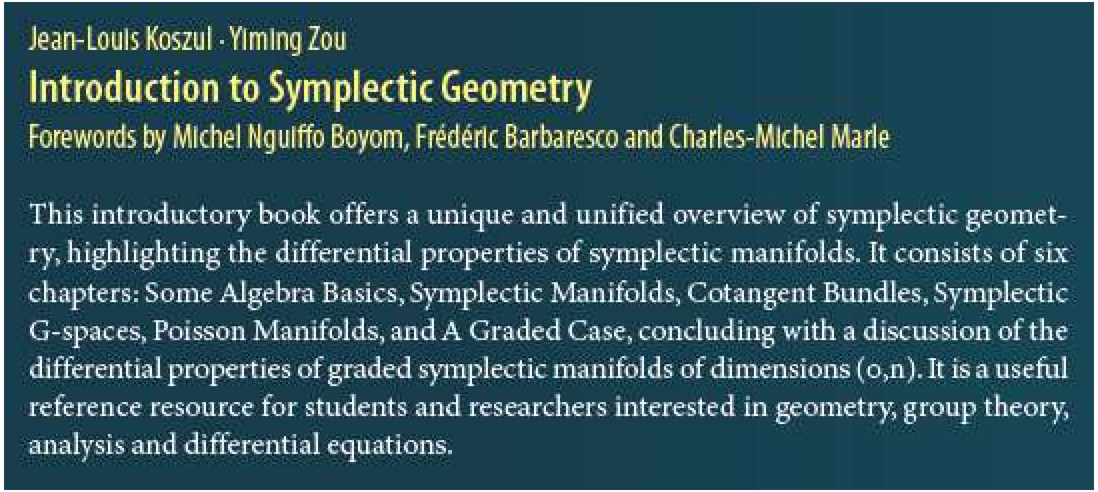
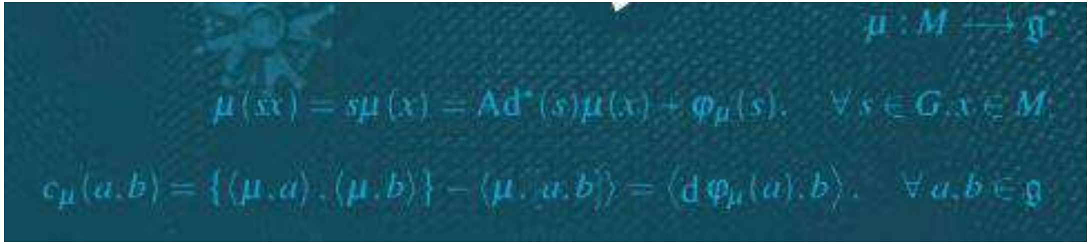


Jean-Louis Koszul · Yiming Zou

Introduction to Symplectic Geometry

Forewords by Michel Nguiffo Boyom, Frédéric Barbaresco and Charles-Michel Marle

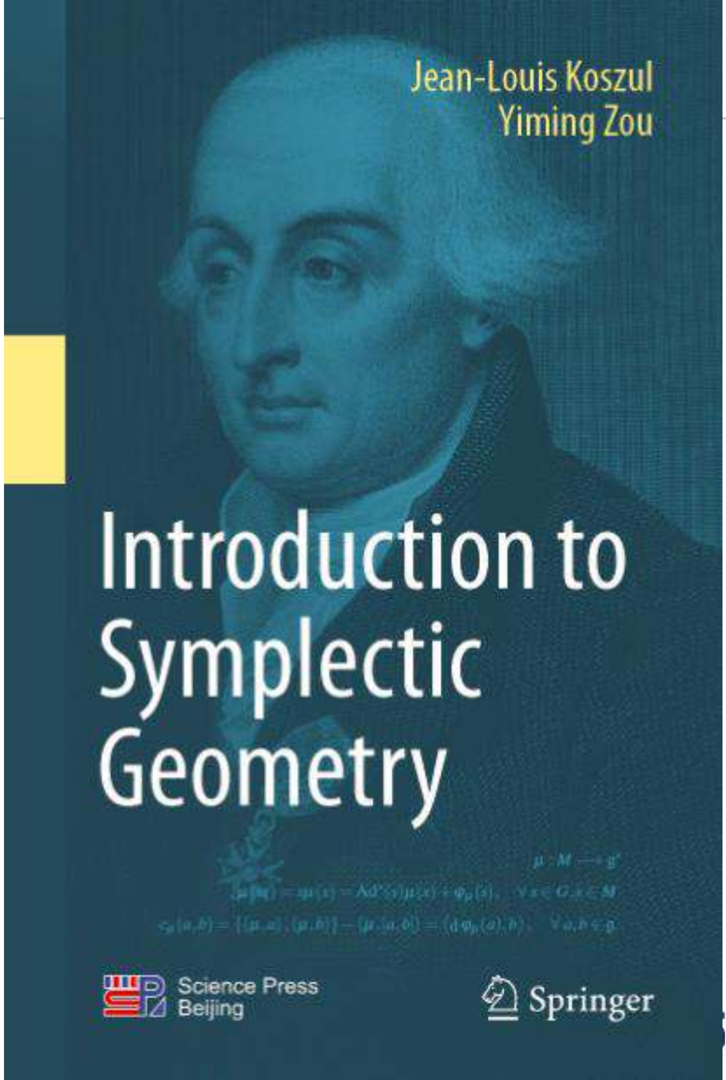
This introductory book offers a unique and unified overview of symplectic geometry, highlighting the differential properties of symplectic manifolds. It consists of six chapters: Some Algebra Basics, Symplectic Manifolds, Cotangent Bundles, Symplectic G-spaces, Poisson Manifolds, and A Graded Case, concluding with a discussion of the differential properties of graded symplectic manifolds of dimensions (0,n). It is a useful reference resource for students and researchers interested in geometry, group theory, analysis and differential equations.



$$\mu : M \longrightarrow \mathfrak{g}^*$$

$$\mu(sx) = s\mu(x) = \mathrm{Ad}^*(s)\mu(x) + \varphi_\mu(s), \quad \forall s \in G, x \in M$$

$$c_\mu(a,b) = \{\langle\mu,a\rangle,\langle\mu,b\rangle\} - \langle\mu,[a,b]\rangle = \langle \mathrm{d}\varphi_\mu(a), b\rangle, \quad \forall a,b \in \mathfrak{g}$$



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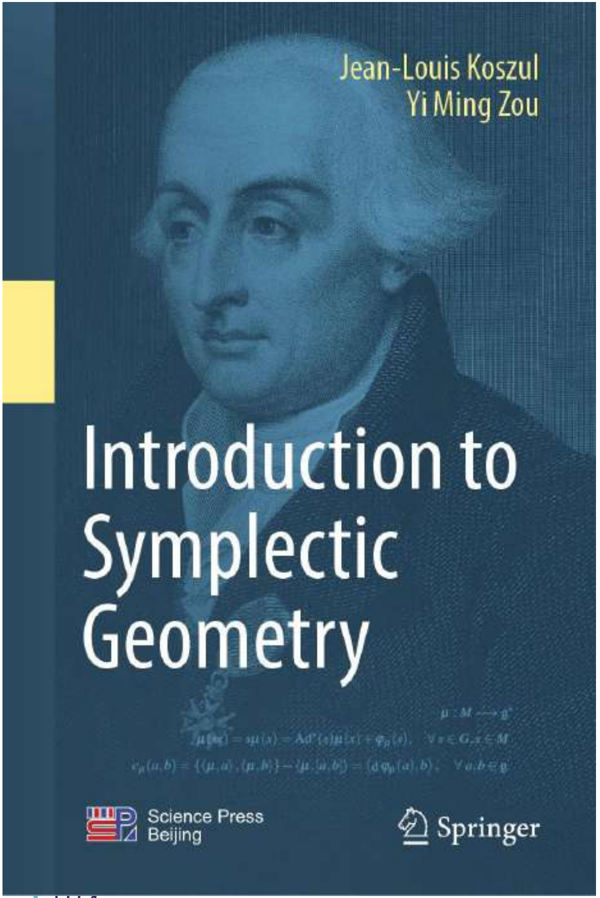

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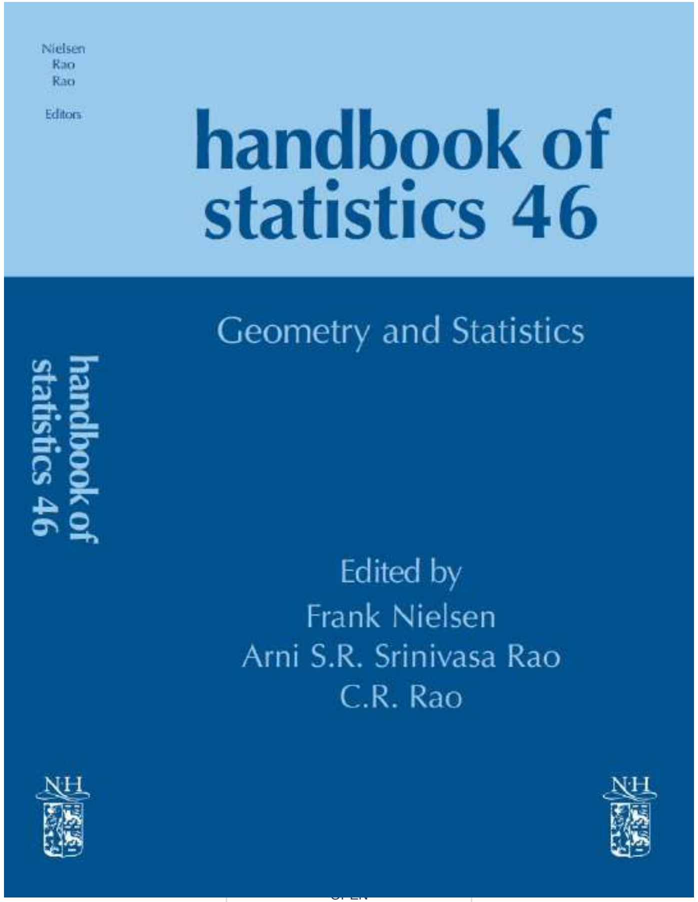

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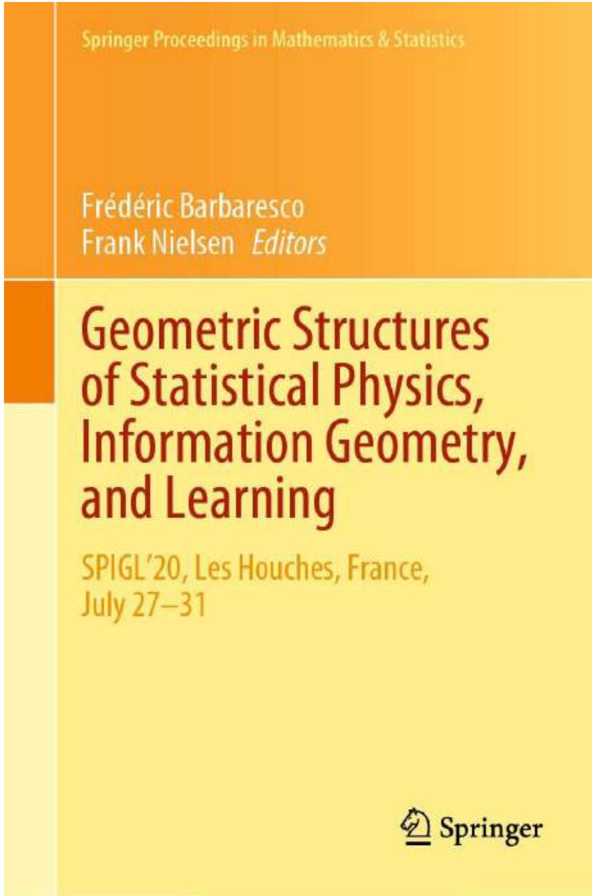

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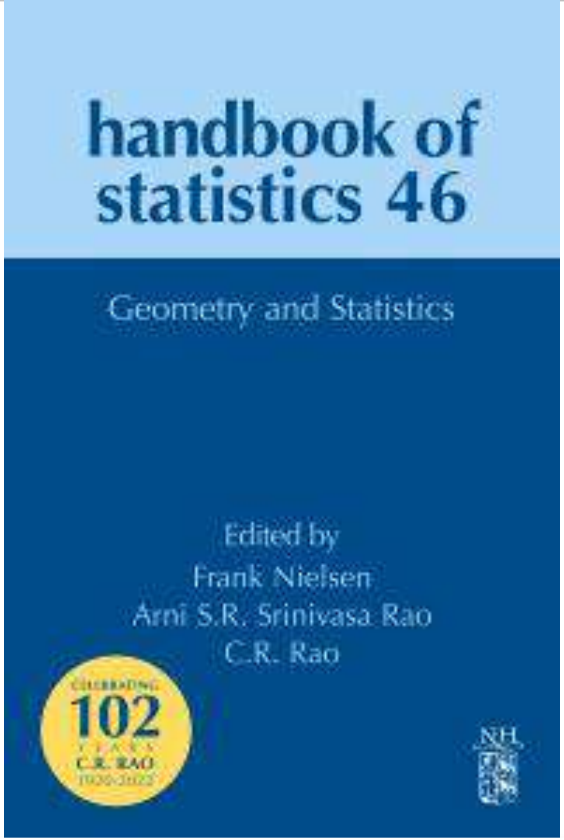

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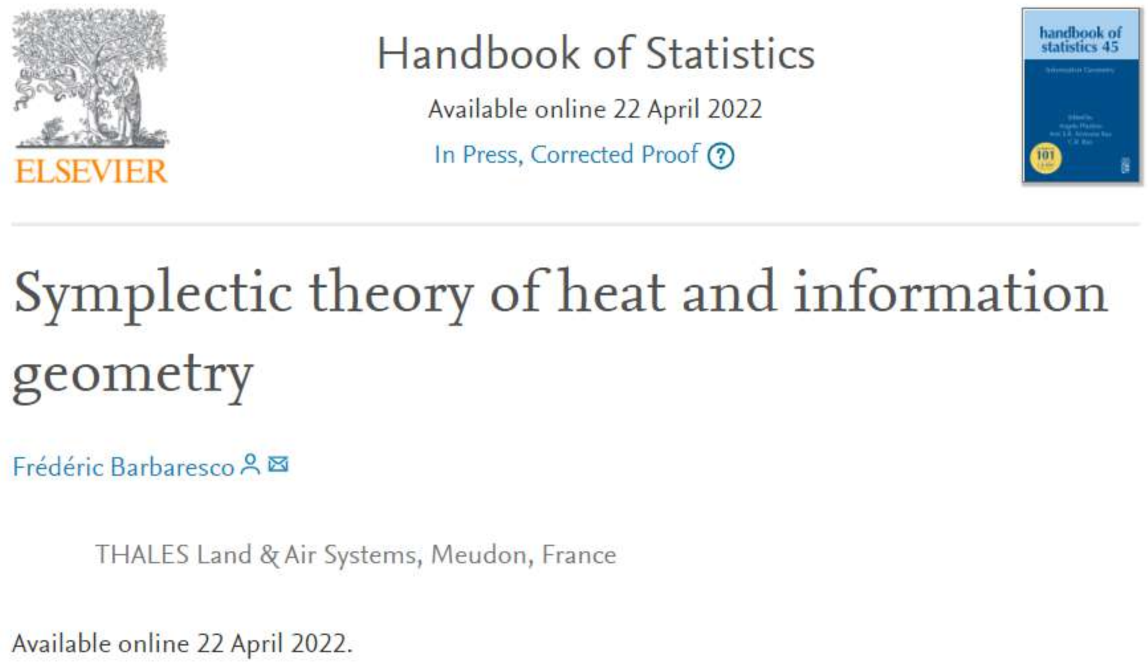

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Frédéric Barbaresco

THALES Land & Air Systems, Meudon, France

Available online 22 April 2022.

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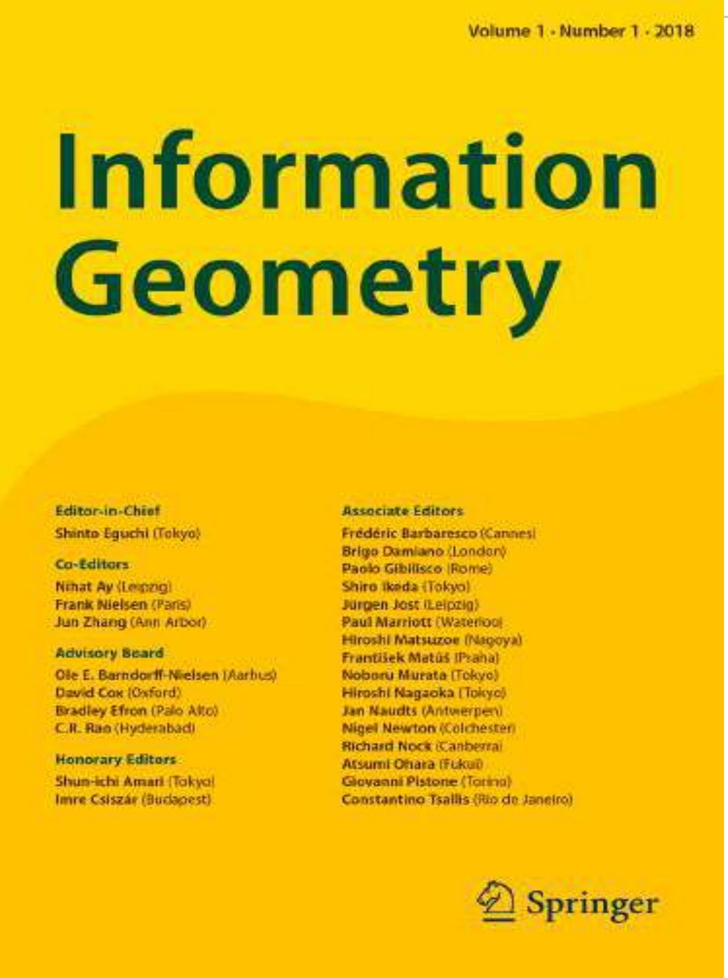
INFORMATION GEOMETRY, SPRINGER JOURNAL

https://www.springer.com/mathematics/geometry/journal/41884



Volume 1 · Number 1 · 2018

Information Geometry

Springer

Special Issue:
Affine Differential Geometry and Hesse Geometry:
A Tribute and Memorial to Jean-Louis Koszul
Submission Deadline: 30th November 2019

Jean-Louis Koszul (January 3, 1921 – January 12, 2018) was a French mathematician with prominent influence to a wide range of mathematical fields. He was a second generation member of Bourbaki, with several notions in geometry and algebra named after him. He made a great contribution to the fundamental theory of Differential Geometry, which is foundation of Information Geometry. The special issue is dedicated to Koszul for the mathematics he developed that bear on information sciences.
Both original contributions and review articles are solicited. Topics include but are not limited to:

- Affine differential geometry over statistical manifolds
- Hessian and Kahler geometry
- Divergence geometry
- Convex geometry and analysis
- Differential geometry over homogeneous and symmetric spaces
- Jordan algebras and graded Lie algebras
- Pre-Lie algebras and their cohomology
- Geometric mechanics and Thermodynamics over homogeneous spaces

Guest Editor:
Hideyuki Ishi (Graduate School of Mathematics, Nagoya University)

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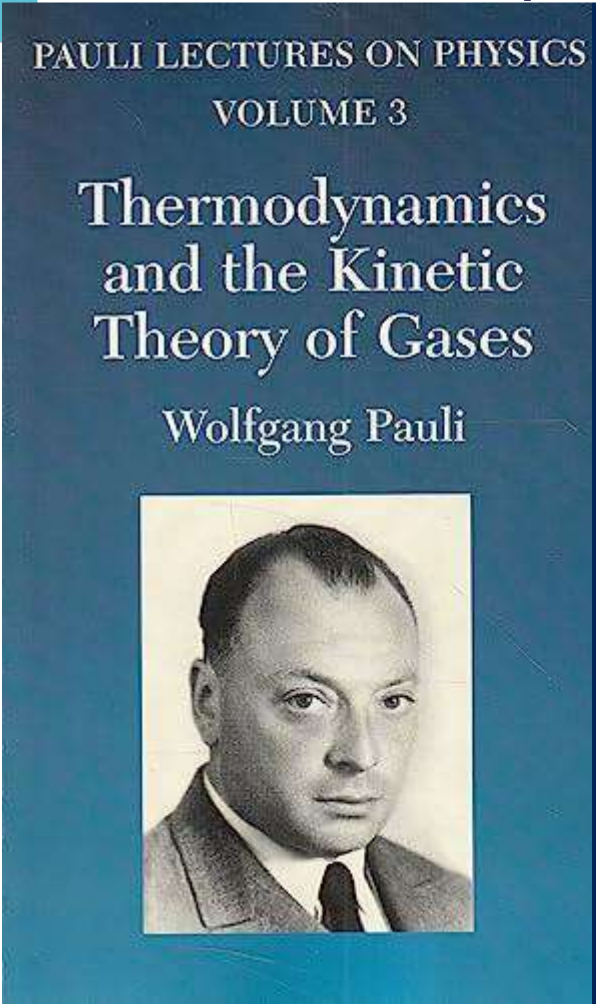

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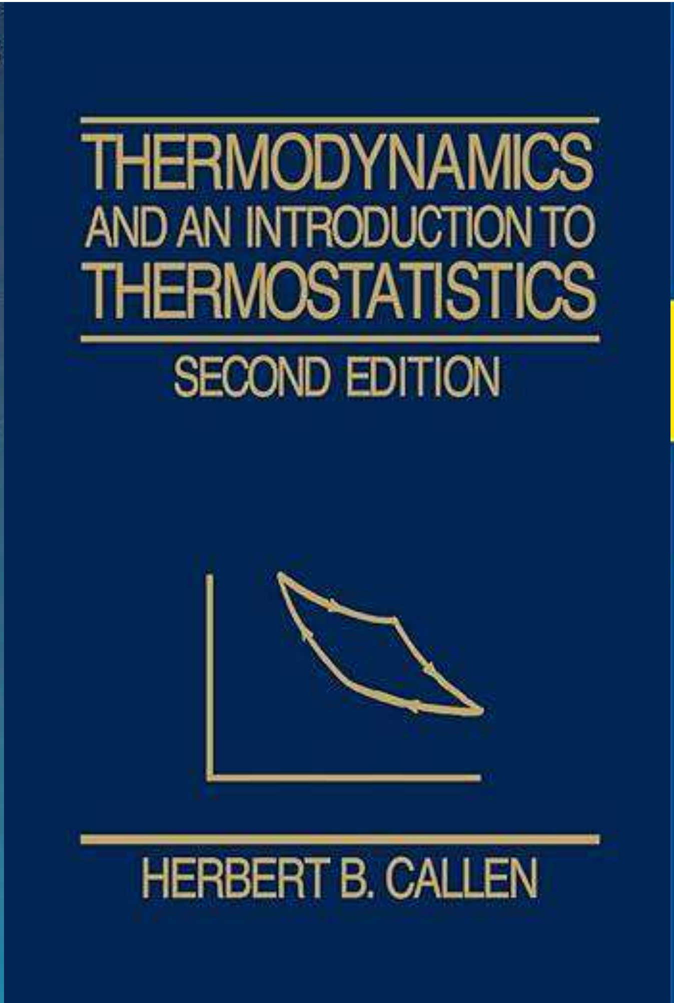

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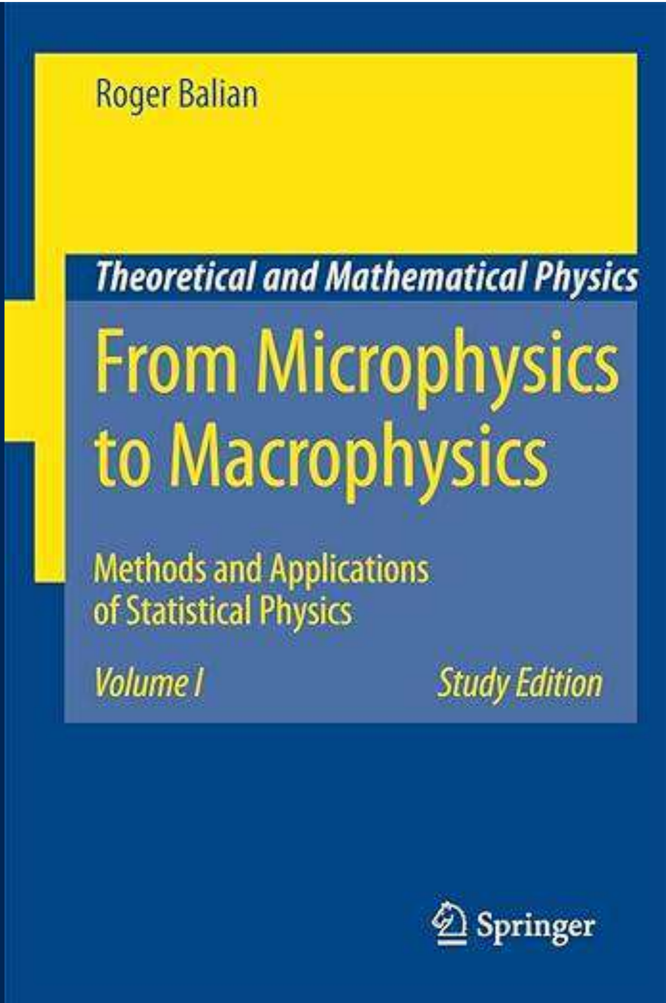

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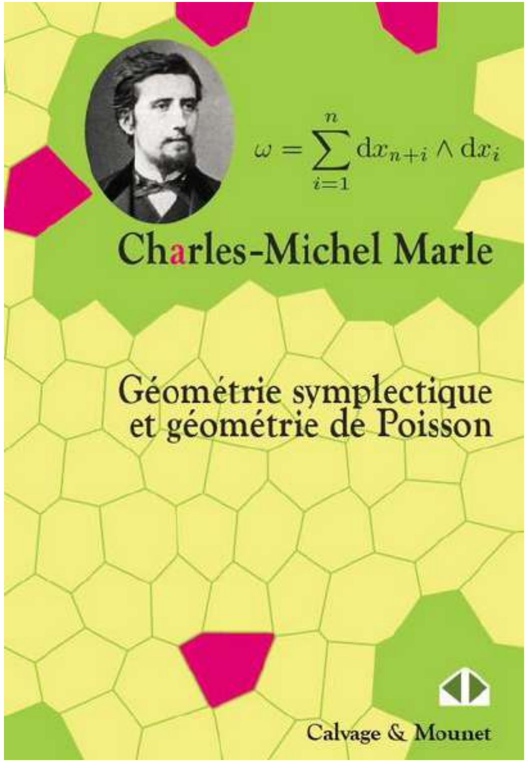

$$\omega = \sum_{i=1}^{n} \mathrm{d}x_{n+i} \wedge \mathrm{d}x_i$$

Charles-Michel Marle

Géométrie symplectique
et géométrie de Poisson

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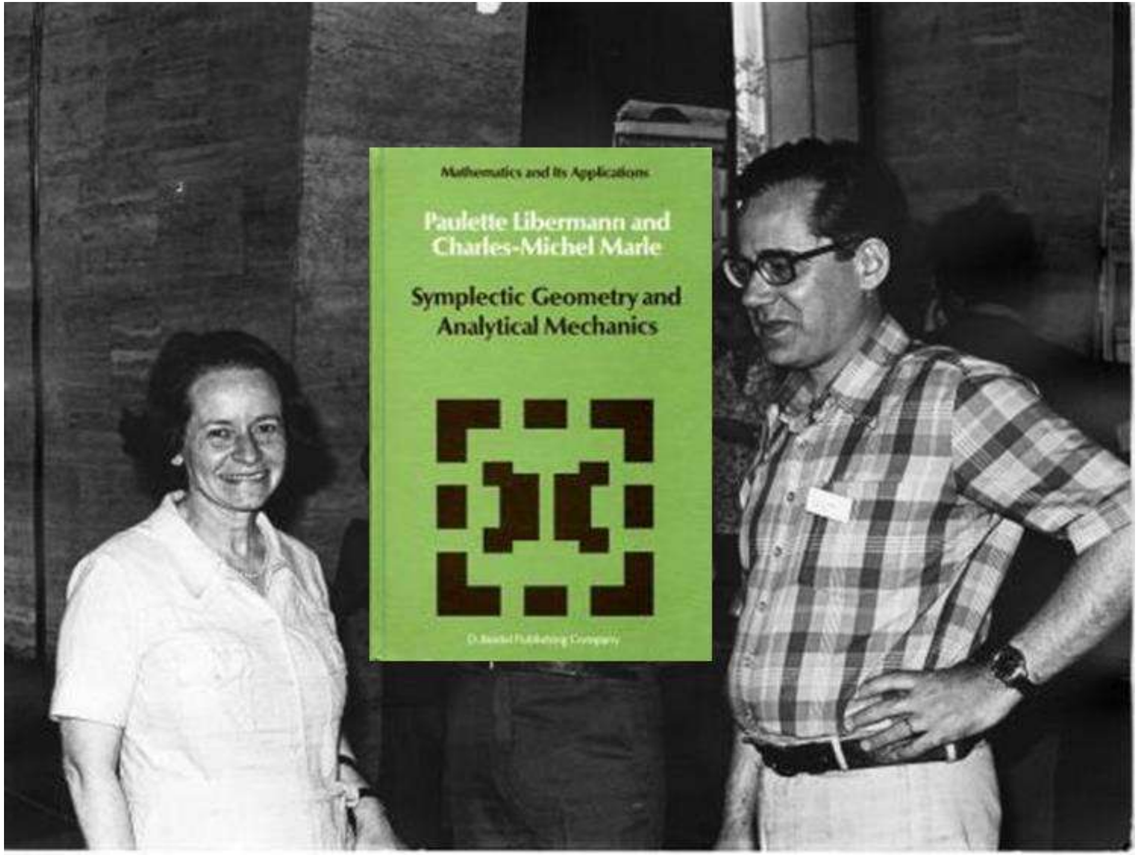
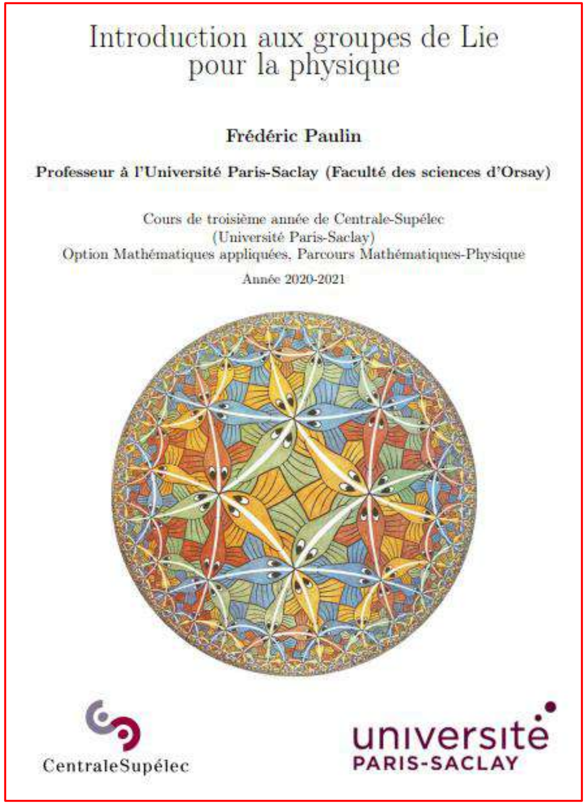


Introduction to Lie Groups Representation Theory , Symplectic and Poisson Geometries (Université Paris Saclay & Sorbonne Université Lectures)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pour la physique

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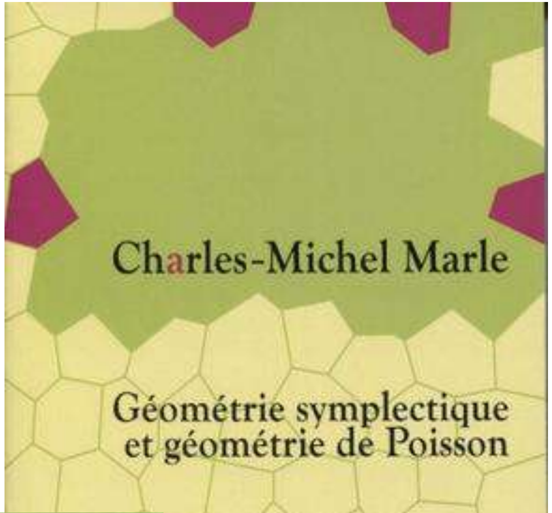
Jean-Bernard Zuber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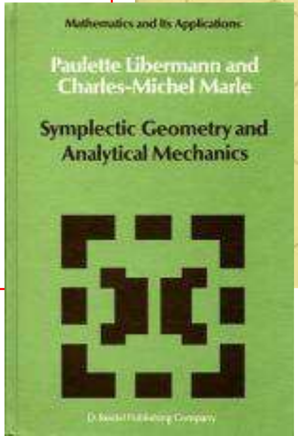
Géométrie symplectique
et géométrie de Poisson

Calvage & Mounet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Poisson Geometry

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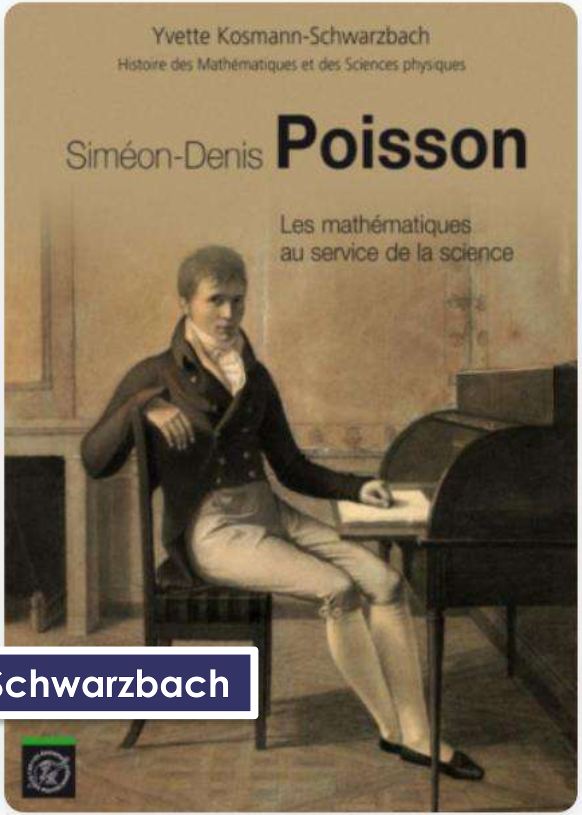

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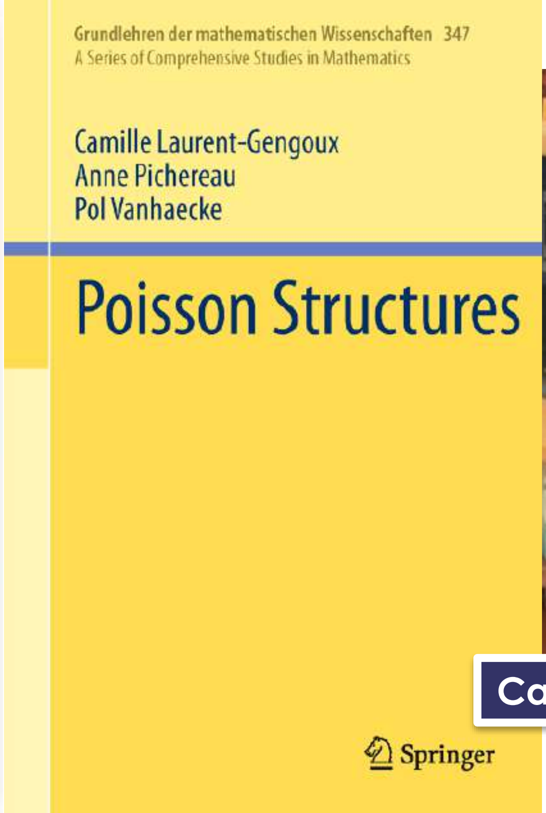
Histoire des Mathématiques et des Sciences physiques

Siméon-Denis **Poisson**

Les mathématiques
au service de la science



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Organisateurs

- **Charles Ehresmann**, University of Strasbourg, France.
- **André Lichnerowicz**, Collège de France.

Participants

- E. Bampiani, Roma, Italy.
- S. S. Chern, Chicago, USA.
- E. T. Davies, Southampton, England.
- P. Dedecker, Brussels, Belgium.
- B. Eckmann, Zürich, Switzerland.
- E. Heinz, Göttingen, Germany.
- N. H. Kuiper, Wageningen, Netherlands.
- H. Rund, Bonn, Germany.
- M. Villa, Bologne, Italy.
- T. J. Willmore, Durham, England.
- **J. L. Koszul**, Strassbourg, France.
- **Paulette Libermann**, Strasbourg, France.
- **G. Reeb**, Grenoble, France.
- L. Schwartz, Paris, France.
- **J. M. Souriau**, Tunis, France.
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- **G. de Rham**, Lausanne, Switzerland.
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- H. B. Shutrick, Liverpool, England.
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- W. Süss, Freiburg, Germany.
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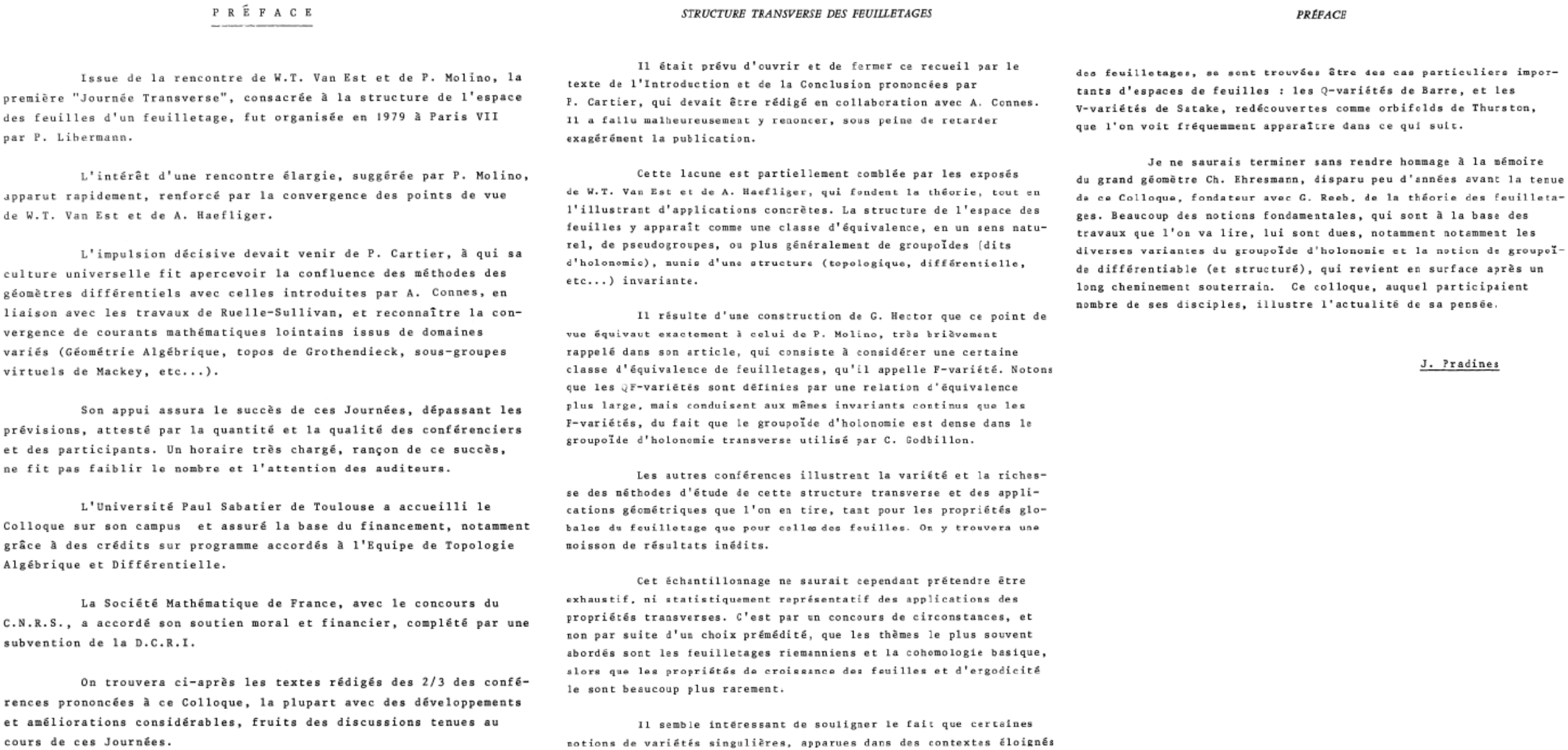
Transverse Structure of Foliations



PRÉFACE

Issue de la rencontre de W.T. Van Est et de P. Molino, la première "Journée Transverse", consacrée à la structure de l'espace des feuilles d'un feuilletage, fut organisée en 1979 à Paris VII par P. Libermann.

L'intérêt d'une rencontre élargie, suggérée par P. Molino, apparut rapidement, renforcé par la convergence des points de vue de W.T. Van Est et de A. Haefliger.

L'impulsion décisive devait venir de P. Cartier, à qui sa culture universelle fit apercevoir la confluence des méthodes des géomètres différentiels avec celles introduites par A. Connes, en liaison avec les travaux de Ruelle-Sullivan, et reconnaître la convergence de courants mathématiques lointains issus de domaines variés (Géométrie Algébrique, topos de Grothendieck, sous-groupes virtuels de Mackey, etc...).

Son appui assura le succès de ces Journées, dépassant les prévisions, attesté par la quantité et la qualité des conférenciers et des participants. Un horaire très chargé, rançon de ce succès, ne fit pas faiblir le nombre et l'attention des auditeurs.

L'Université Paul Sabatier de Toulouse a accueilli le Colloque sur son campus et assuré la base du financement, notamment grâce à des crédits sur programme accordés à l'Equipe de Topologie Algébrique et Différentielle.

La Société Mathématique de France, avec le concours du C.N.R.S., a accordé son soutien moral et financier, complété par une subvention de la D.C.R.I.

On trouvera ci-après les textes rédigés des 2/3 des conférences prononcées à ce Colloque, la plupart avec des développements et améliorations considérables, fruits des discussions tenues au cours de ces Journées.

Il était prévu d'ouvrir et de fermer ce recueil par le texte de l'Introduction et de la Conclusion prononcées par P. Cartier, qui devait être rédigé en collaboration avec A. Connes. Il a fallu malheureusement y renoncer, sous peine de retarder exagérément la publication.

Cette lacune est partiellement comblée par les exposés de W.T. Van Est et de A. Haefliger, qui fondent la théorie, tout en l'illustrant d'applications concrètes. La structure de l'espace des feuilles y apparaît comme une classe d'équivalence, en un sens naturel, de pseudogroupes, ou plus généralement de groupoïdes (dits d'holonomie), munis d'une structure (topologique, différentielle, etc...) invariante.

Il résulte d'une construction de G. Hector que ce point de vue équivaut exactement à celui de P. Molino, très brièvement rappelé dans son article, qui consiste à considérer une certaine classe d'équivalence de feuilletages, qu'il appelle F-variété. Notons que les QF-variétés sont définies par une relation d'équivalence plus large, mais conduisent aux mêmes invariants continus que les F-variétés, du fait que le groupoïde d'holonomie est dense dans le groupoïde d'holonomie transverse utilisé par C. Godbillon.

Les autres conférences illustrent la variété et la richesse des méthodes d'étude de cette structure transverse et des applications géométriques que l'on en tire, tant pour les propriétés globales du feuilletage que pour celles des feuilles. On y trouvera une moisson de résultats inédits.

Cet échantillonnage ne saurait cependant prétendre être exhaustif, ni statistiquement représentatif des applications des propriétés transverses. C'est par un concours de circonstances, et non par suite d'un choix prémédité, que les thèmes le plus souvent abordés sont les feuilletages riemanniens et la cohomologie basique, alors que les propriétés de croissance des feuilles et d'ergodicité le sont beaucoup plus rarement.

Il semble intéressant de souligner le fait que certaines notions de variétés singulières, apparues dans des contextes éloignés des feuilletages, se sont trouvées être des cas particuliers importants d'espaces de feuilles : les Q-variétés de Barre, et les V-variétés de Satake, redécouvertes comme orbifolds de Thurston, que l'on voit fréquemment apparaître dans ce qui suit.

Je ne saurais terminer sans rendre hommage à la mémoire du grand géomètre Ch. Ehresmann, disparu peu d'années avant la tenue de ce Colloque, fondateur avec G. Reeb, de la théorie des feuilletages. Beaucoup des notions fondamentales, qui sont à la base des travaux que l'on va lire, lui sont dues, notamment notamment les diverses variantes du groupoïde d'holonomie et la notion de groupoïde différentiable (et structuré), qui revient en surface après un long cheminement souterrain. Ce colloque, auquel participaient nombre de ses disciples, illustre l'actualité de sa pensée.

J. Pradines

Transverse Structure of Foliations

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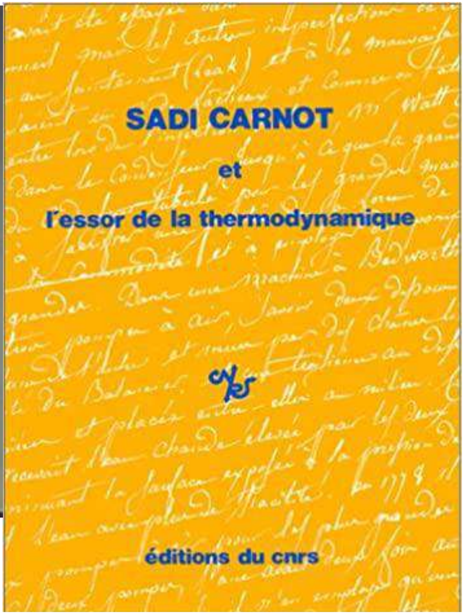
150th birthday of Thermodynamics – Ecole Polytechnique



SADI CARNOT
et
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RÉFLEXIONS
SUR LA
PUISSANCE MOTRICE
DU FEU
ET
SUR LES MACHINES
PROPRES A DÉVELOPPER CETTE PUISSANCE.
PAR S. CARNOT,
ANCIEN ÉLÈVE DE L'ÉCOLE POLYTECHNIQUE.

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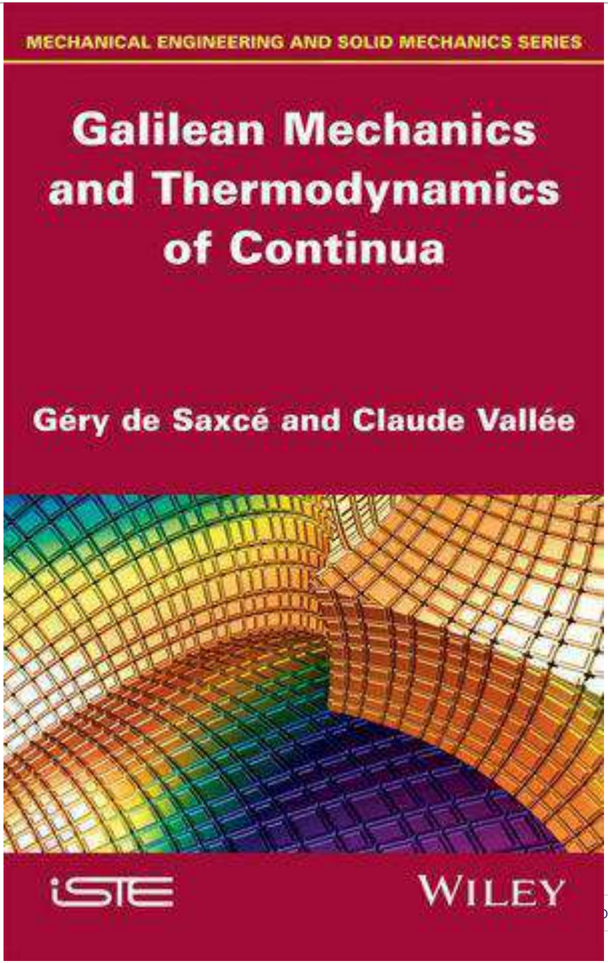
1st & 2nd Thermodynamics Principles

1824-2024: 200 years publication of Carnot's Book

Gery de Saxcé & Claude Vallée



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Galilean Mechanics and Thermodynamics of Continua

Géry de Saxcé and Claude Vallée

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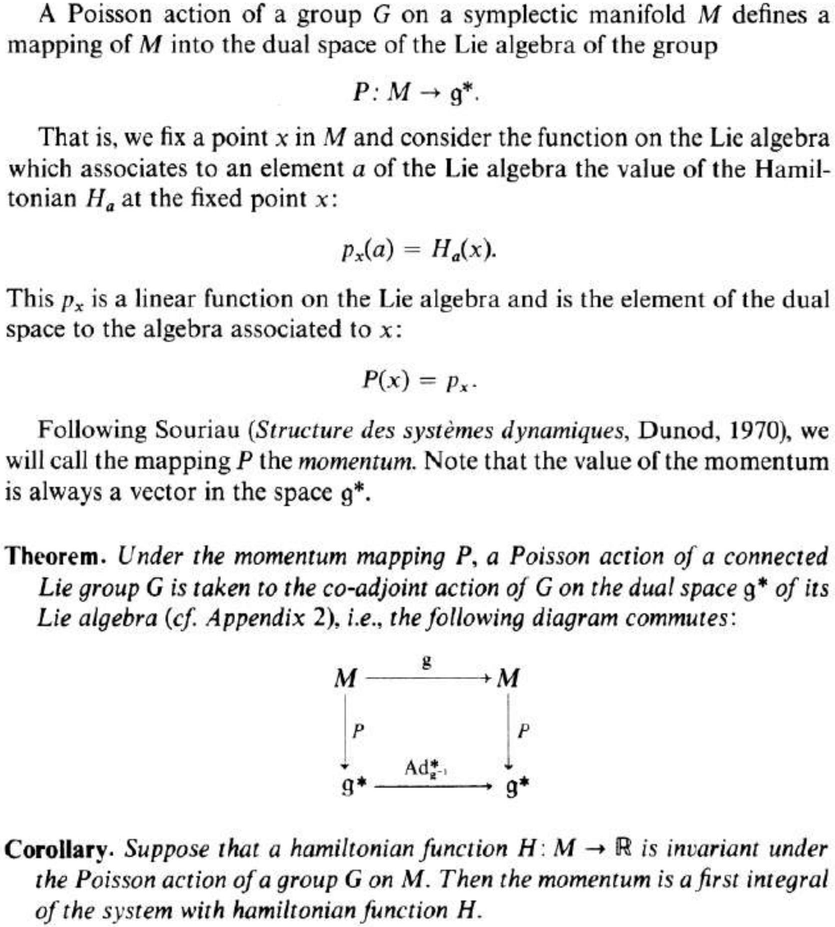
This title proposes a unified approach to continuum mechanics which is consistent with Galilean relativity. Based on the notion of affine tensors, a simple generalization of the classical tensors, this approach allows gathering the usual mechanical entities — mass, energy, force, moment, stresses, linear and angular momentum — in a single tensor.

Starting with the basic subjects, and continuing through to the most advanced topics, the authors' presentation is progressive, inductive and bottom-up. They begin with the concept of an affine tensor, a natural extension of the classical tensors. The simplest types of affine tensors are the points of an affine space and the affine functions on this space, but there are more complex ones which are relevant for mechanics − torsors and momenta. The essential point is to derive the balance equations of a continuum from a unique principle which claims that these tensors are affine-divergence free.

https://www.wiley.com/en-us/Galilean+Mechanics+and+Thermodynamics+of+Continua-p-9781848216426

SOURIAU'S MOMENT MAP
By VLADIMIR ARNOLD



A Poisson action of a group G on a symplectic manifold M defines a mapping of M into the dual space of the Lie algebra of the group

$$P: M \to \mathfrak{g}^*.$$

That is, we fix a point x in M and consider the function on the Lie algebra which associates to an element a of the Lie algebra the value of the Hamiltonian H_a at the fixed point x:

$$p_x(a) = H_a(x).$$

This p_x is a linear function on the Lie algebra and is the element of the dual space to the algebra associated to x:

$$P(x) = p_x.$$

Following Souriau (*Structure des systèmes dynamiques*, Dunod, 1970), we will call the mapping P the *momentum*. Note that the value of the momentum is always a vector in the space $\mathfrak{g}^*$.

Theorem. *Under the momentum mapping P, a Poisson action of a connected Lie group G is taken to the co-adjoint action of G on the dual space $\mathfrak{g}^*$ of its Lie algebra (cf. Appendix 2), i.e., the following diagram commutes:*

$$\begin{tikzcd}
M \arrow[r, "g"] \arrow[d, "P"'] & M \arrow[d, "P"] \\
\mathfrak{g}^* \arrow[r, "\mathrm{Ad}^*_{g^{-1}}"] & \mathfrak{g}^*
\end{tikzcd}$$

Corollary. *Suppose that a hamiltonian function $H: M \to \mathbb{R}$ is invariant under the Poisson action of a group G on M. Then the momentum is a first integral of the system with hamiltonian function H.*

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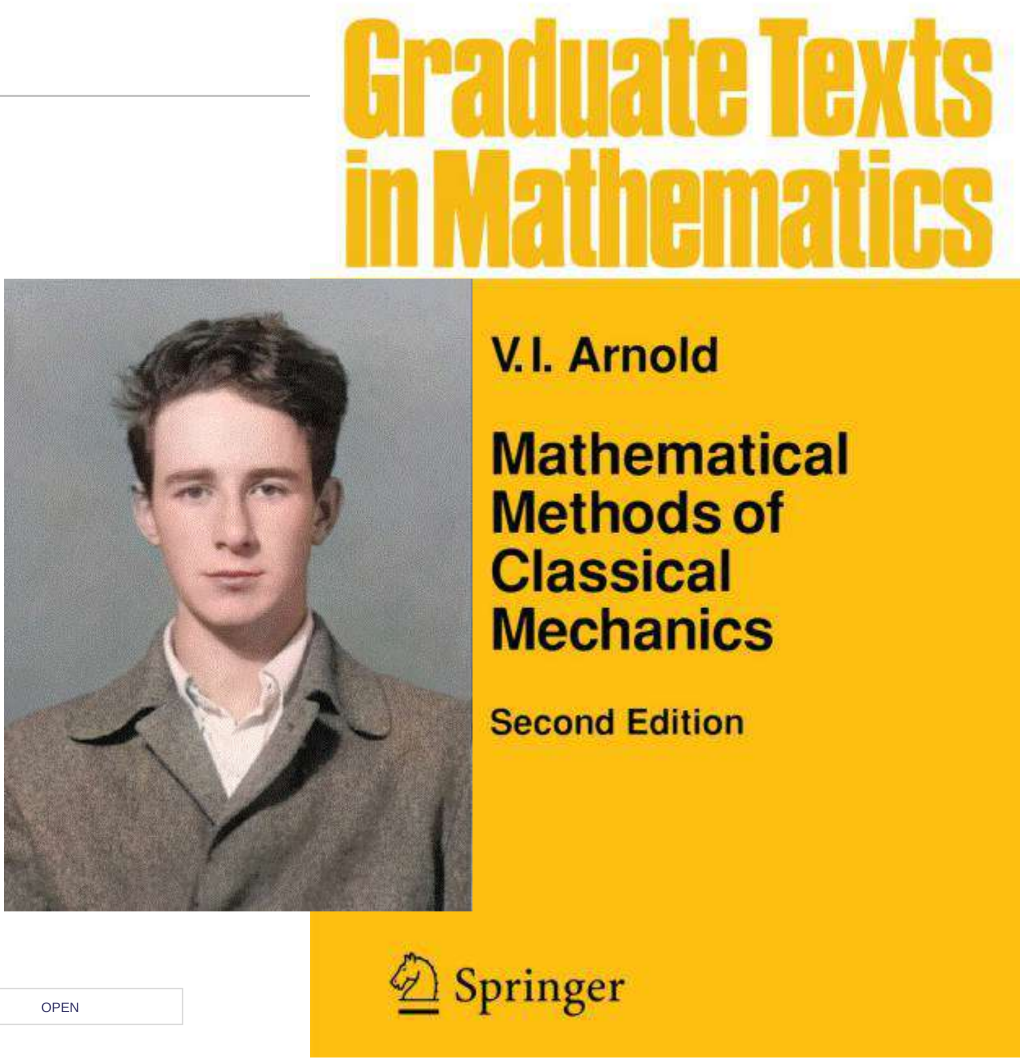

Graduate Texts in Mathematics

V. I. Arnold

Mathematical Methods of Classical Mechanics

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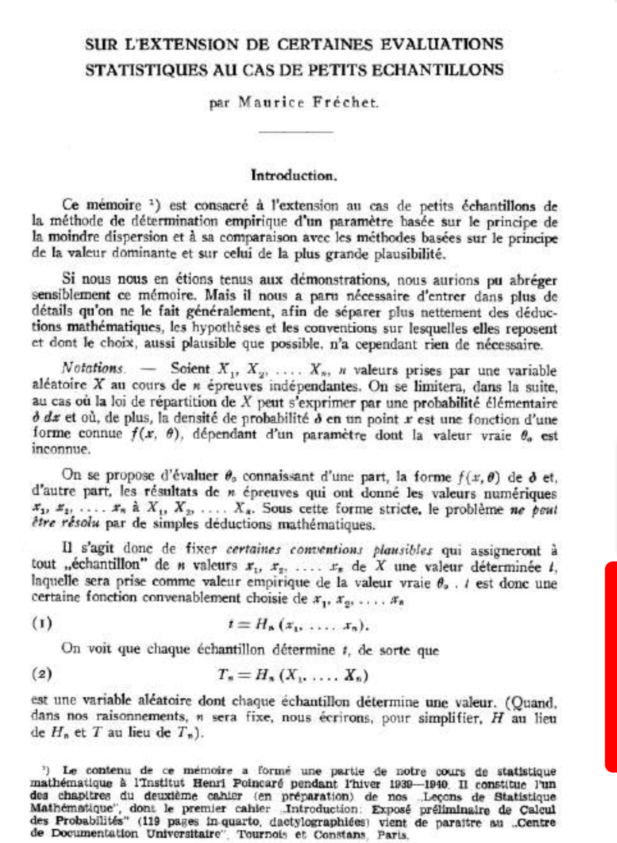
Springer

Maurice Fréchet, papier de 1943 et cours de l'IHP de l'Hiver 1939

M. Fréchet, Sur l'extension de certaines evaluations statistiques au cas de petits échantillons, Revue de l'Institut International de Statistique, Vol. 11, No. 3/4 (1943), pp. 182-205



SUR L'EXTENSION DE CERTAINES EVALUATIONS STATISTIQUES AU CAS DE PETITS ECHANTILLONS

par Maurice Fréchet.

Introduction.

Ce mémoire [1]) est consacré à l'extension au cas de petits échantillons de la méthode de détermination empirique d'un paramètre basée sur le principe de la moindre dispersion et à sa comparaison avec les méthodes basées sur le principe de la valeur dominante et sur celui de la plus grande plausibilité.

Si nous nous en étions tenus aux démonstrations, nous aurions pu abréger sensiblement ce mémoire. Mais il nous a paru nécessaire d'entrer dans plus de détails qu'on ne le fait généralement, afin de séparer plus nettement des déductions mathématiques, les hypothèses et les conventions sur lesquelles elles reposent et dont le choix, aussi plausible que possible, n'a cependant rien de nécessaire.

Notations. — Soient $X_1, X_2, \ldots. X_n$, n valeurs prises par une variable aléatoire X au cours de n épreuves indépendantes. On se limitera, dans la suite, au cas où la loi de répartition de X peut s'exprimer par une probabilité élémentaire $\delta\, dx$ et où, de plus, la densité de probabilité δ en un point x est une fonction d'une forme connue $f(x, \theta)$, dépendant d'un paramètre dont la valeur vraie θ_0 est inconnue.

On se propose d'évaluer θ_0 connaissant d'une part, la forme $f(x,\theta)$ de δ et, d'autre part, les résultats de n épreuves qui ont donné les valeurs numériques $x_1, x_2, \ldots. x_n$ à $X_1, X_2, \ldots. X_n$. Sous cette forme stricte, le problème *ne peut être résolu* par de simples déductions mathématiques.

Il s'agit donc de fixer *certaines conventions plausibles* qui assigneront à tout „échantillon" de n valeurs $x_1, x_2, \ldots. x_n$ de X une valeur déterminée t, laquelle sera prise comme valeur empirique de la valeur vraie θ_0 . t est donc une certaine fonction convenablement choisie de $x_1, x_2, \ldots. x_n$

$$t = H_n(x_1, \ldots. x_n). \tag{1}$$

On voit que chaque échantillon détermine t, de sorte que

$$T_n = H_n(X_1, \ldots. X_n) \tag{2}$$

est une variable aléatoire dont chaque échantillon détermine une valeur. (Quand, dans nos raisonnements, n sera fixe, nous écrirons, pour simplifier, H au lieu de H_n et T au lieu de T_n).

---

[1]) Le contenu de ce mémoire a formé une partie de notre cours de statistique mathématique à l'Institut Henri Poincaré pendant l'hiver 1939—1940. Il constitue l'un des chapitres du deuxième cahier (en préparation) de nos „Leçons de Statistique Mathématique", dont le premier cahier „Introduction: Exposé préliminaire de Calcul des Probabilités" (119 pages in-quarto, dactylographiées) vient de paraitre au „Centre de Documentation Universitaire", Tournois et Constans, Paris.

Manuscrit perdu du cours de statistique mathématique à l'Institut Henri Poincaré pendant l'Hiver 1939-1940 !

[1]) Le contenu de ce mémoire a formé une partie de notre cours de statistique mathématique à l'Institut Henri Poincaré pendant l'hiver 1939—1940. Il constitue l'un des chapitres du deuxième cahier (en préparation) de nos „Leçons de Statistique Mathématique", dont le premier cahier „Introduction: Exposé préliminaire de Calcul des Probabilités" (119 pages in-quarto, dactylographiées) vient de paraitre au „Centre de Documentation Universitaire", Tournois et Constans, Paris.

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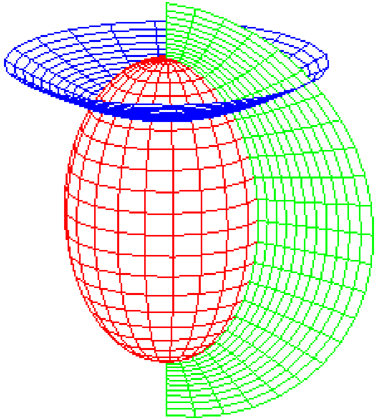
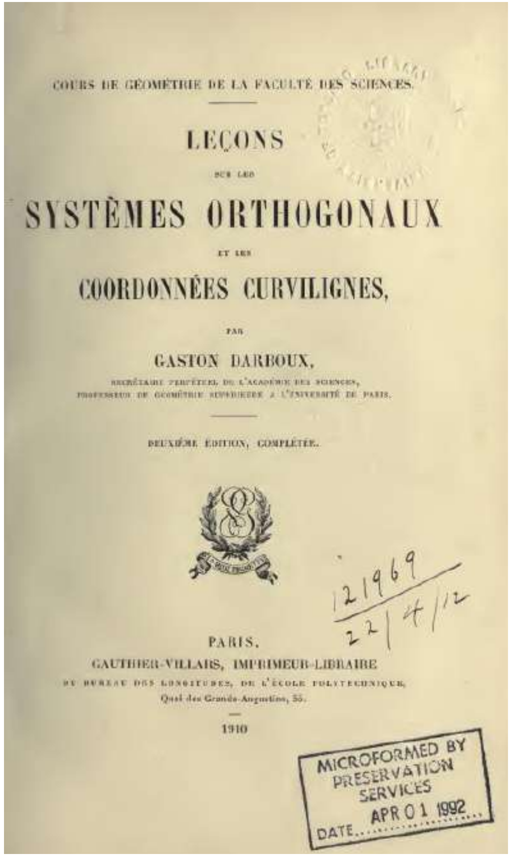
Triple Orthogonal Systems : Darboux and Lamé





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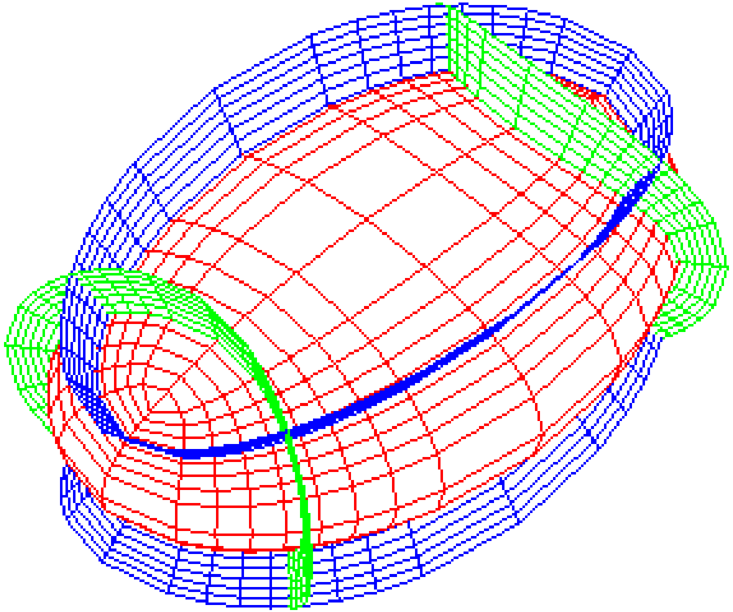
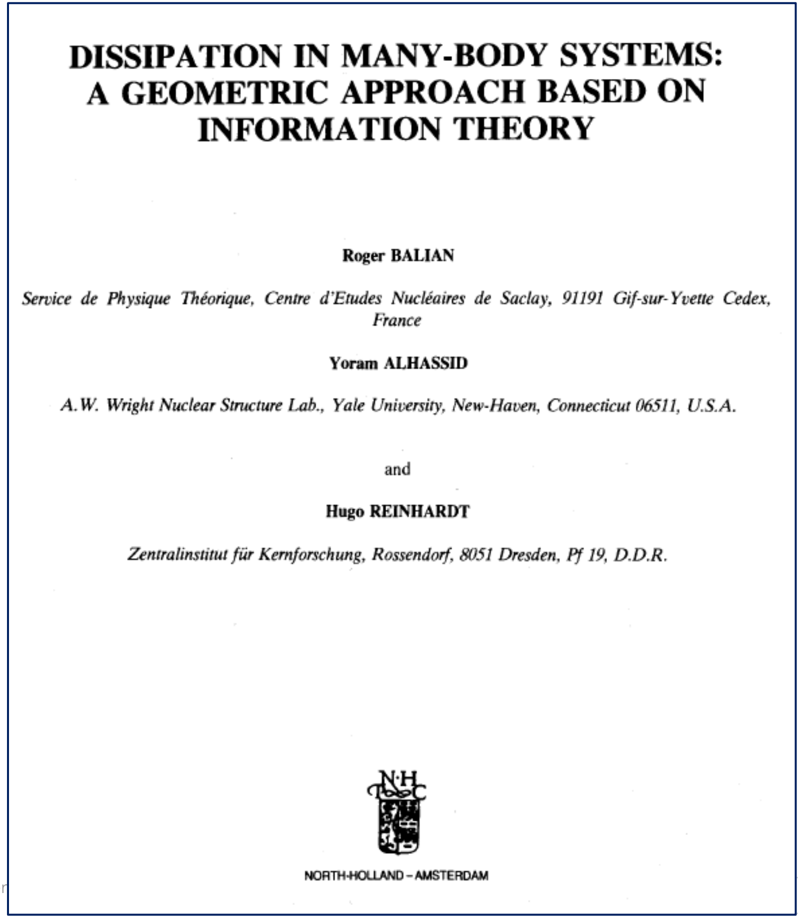


Geometry of Dissipation by Roger BALIAN



DISSIPATION IN MANY-BODY SYSTEMS: A GEOMETRIC APPROACH BASED ON INFORMATION THEORY

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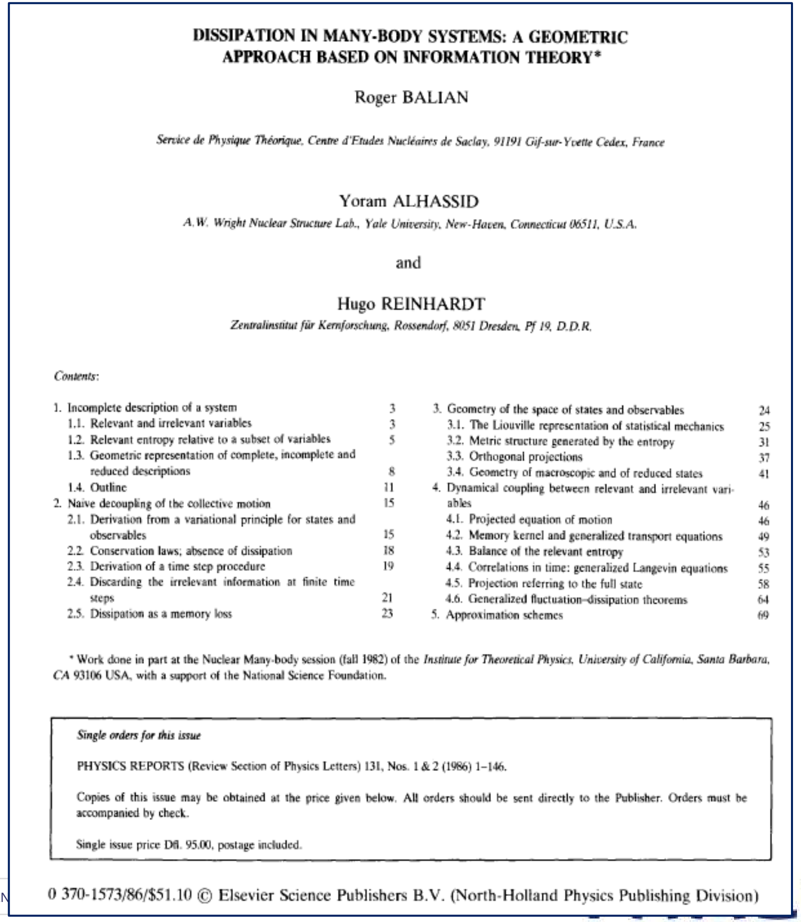

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Contents:

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* Work done in part at the Nuclear Many-body session (fall 1982) of the *Institute for Theoretical Physics, University of California, Santa Barbara*, CA 93106 USA, with a support of the National Science Foundation.

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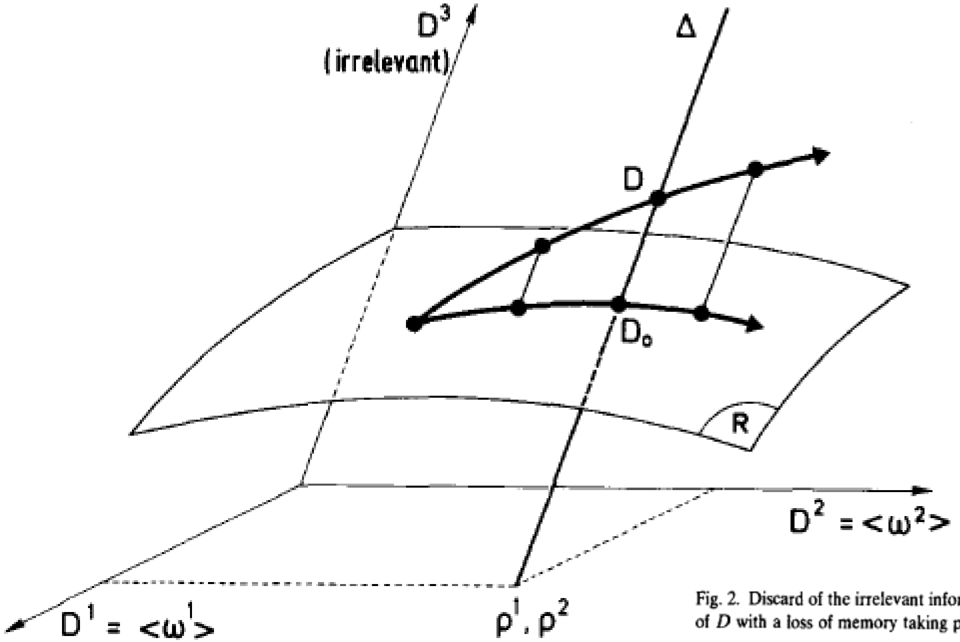


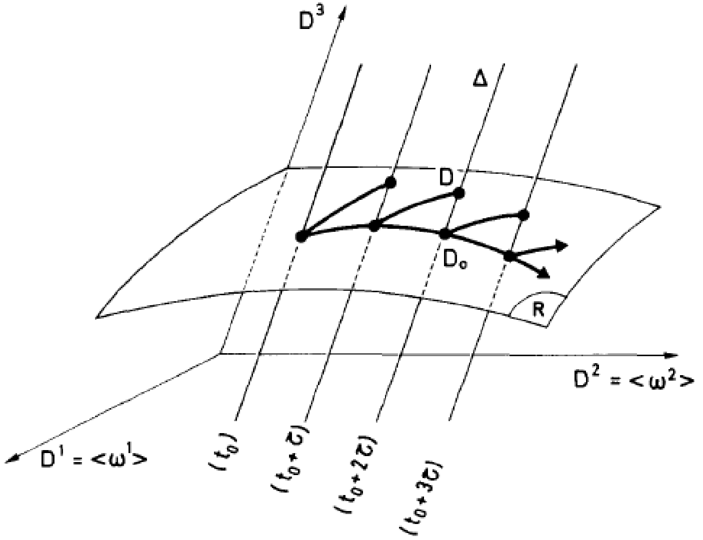


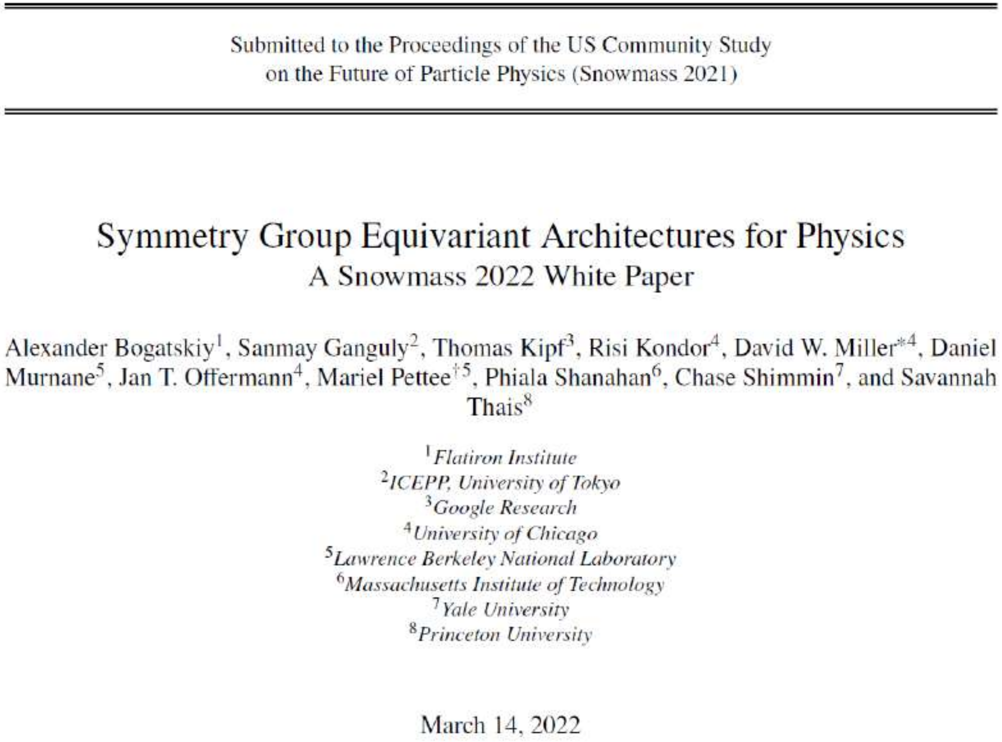
Fig. 2. Discard of the irrelevant information at regular time intervals. An approximate evolution for D_0 is obtained by combining the exact evolution of D with a loss of memory taking place at the times $\tau_0 + n\tau$, and represented by a projection on R.

Fig. 1. Visualization of states in statistical mechanics. Three types of descriptions of a system should be distinguished. (1) A completely specified, *microscopic* state D is represented by a point, the coordinates D^μ of which are the expectation values of a complete set Ω^μ of linearly independent observables. The corresponding entropy $S(D)$ remains constant along a Hamiltonian trajectory. (2) A *macroscopic*, incomplete, description is characterized by the sole knowledge of the expectation values ρ^i of some set ω^i of relevant observables. This defines in the space of microscopic states a plane Δ (here a line), the dimensionality of which is the number of irrelevant variables. (3) The point D_0 of Δ which carries the minimum amount of information lies at the intersection of the plane Δ with the surface R of *reduced* states (having the same dimensionality as the relevant set). It provides for arbitrary observables the least biased predictions compatible with the macroscopic variables ρ. Its entropy $S(D_0)$ is identified with the macroscopic entropy $S(\rho)$ and may exhibit dissipation.

Capture of Symmetries
Symmetry Group Equivariance in Physics

Reference:

- https://arxiv.org/abs/2203.06153



Submitted to the Proceedings of the US Community Study
on the Future of Particle Physics (Snowmass 2021)

Symmetry Group Equivariant Architectures for Physics
A Snowmass 2022 White Paper

Alexander Bogatskiy[1], Sanmay Ganguly[2], Thomas Kipf[3], Risi Kondor[4], David W. Miller[*4], Daniel Murnane[5], Jan T. Offermann[4], Mariel Pettee[†5], Phiala Shanahan[6], Chase Shimmin[7], and Savannah Thais[8]

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[3] *Google Research*
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March 14, 2022